NETWORK ANALYSIS
18EC35

By
K PRABHAVATHI
Assistant Professor
Department of ECE
BGSIT,BG Nagar

SYLLABUS

1. Basic Concepts
2. Network Theorems
3. Transient Behavior and Initial Conditions & Laplace Transforms & Applications
4. Resonance circuits
5. Two port network parameters

Course Outcomes

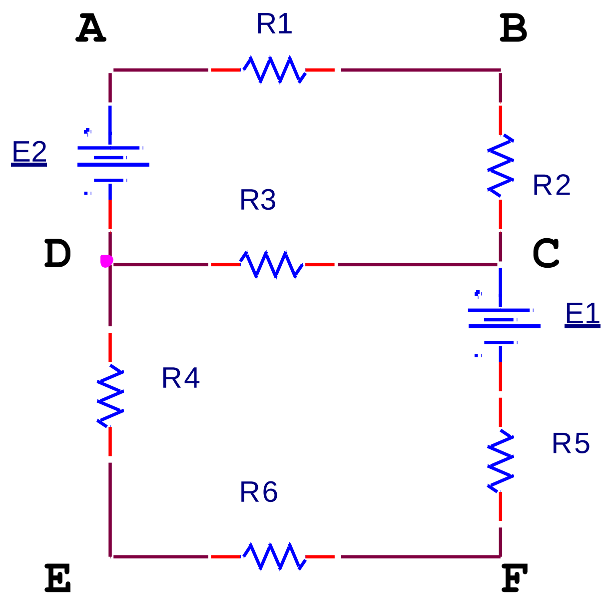
- **Solve electrical circuits using different transformation techniques, mesh and nodal methods.**
- **Analyze complex electric circuits using network theorems.**
- **Evaluate the behavior of R-L-C electrical circuits considering Initial conditions and Laplace transformation.**
- **Analyze series and parallel resonant circuits.**
- **Construct two port models for given network by determining Z, Y, h and T parameters.**

MODULE-1
BASIC CONCEPTS

CONTENTS

INTRODUCTION

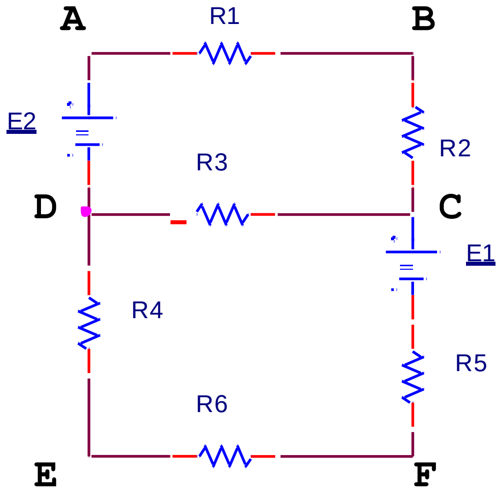
WHAT IS A NETWORK ...?



Any arrangement of various electrical energy sources along with the different circuit elements is called Electrical Network.

Any individual circuit element with two terminals which can be connected to other circuit element is called Network Element.

A part of the network which connects the various points of the network with one another is called A Branch.



Ex: AB, BC, CD, DA, CF, DE & EF

A point where three or more branches meet is called a Junction Point.

Ex: Point C & D

A point at which two or more elements are joined together is called Node.

Ex: A, B, C, D, E, F

A closed path which originates from a particular node, terminating at the same node, traveling through various other nodes, without traveling through any node twice is called Mesh or Loop.

Ex: A-B-C-D-A, A-B-C-F-E-D-A.

Mesh does not contain any other loop within it. A mesh is always a loop but a loop may or may not be a mesh.

Active Network

A network consisting at least one source of energy is called active network.

Ex: Network consisting at least one battery, voltage source, current source,,etc.

Passive Network

A network which contains no energy sources is called passive network.

Ex: Network consisting only elements such as R, L and C without any energy sources.

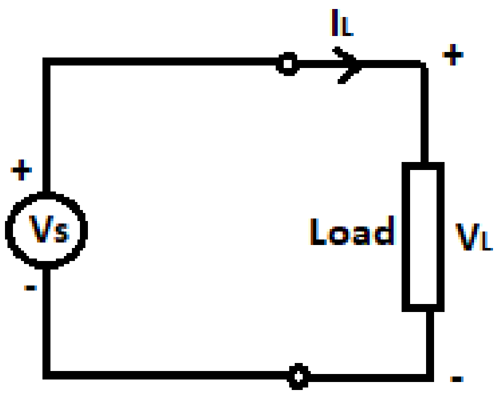
Concept of Ideal & Practical Sources

There are basically two types of energy sources;

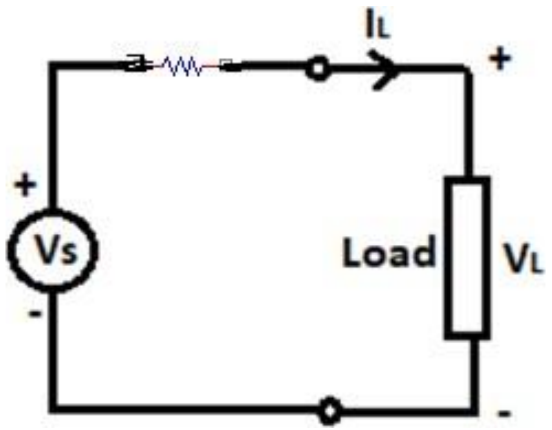
- ✓ Voltage source
- ✓ Current source

These are classified as

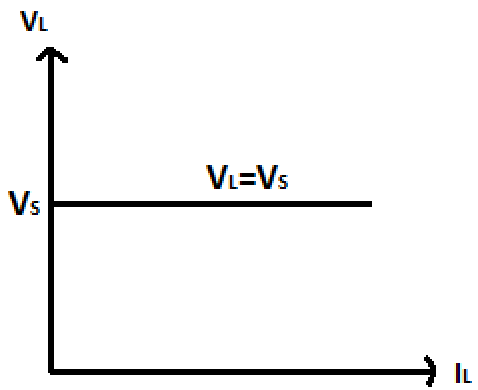
- ➢ Ideal Source
- ➢ Practical source



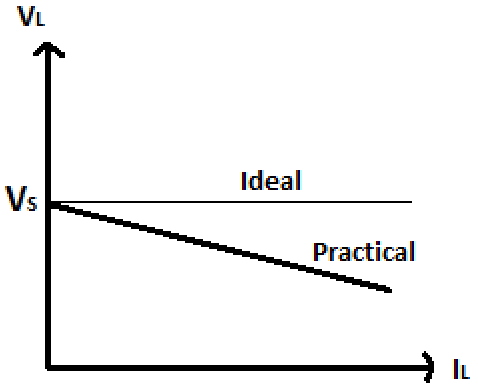
Ideal Voltage Source



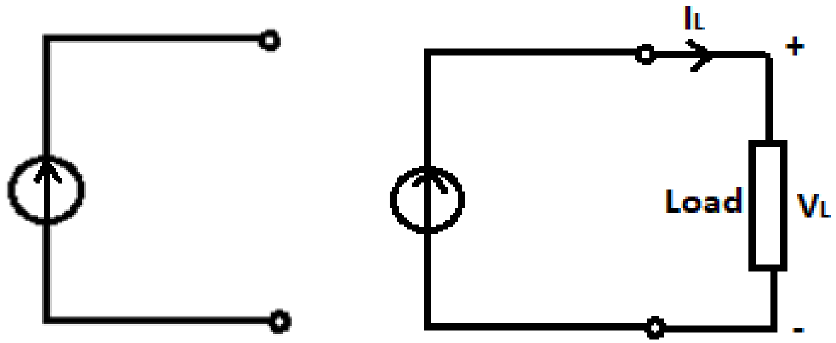
Practical Voltage Source



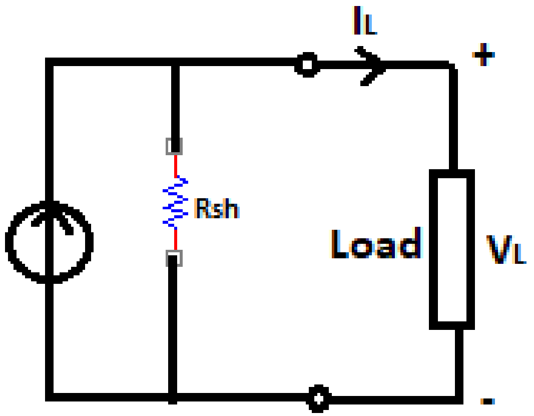
Characteristics



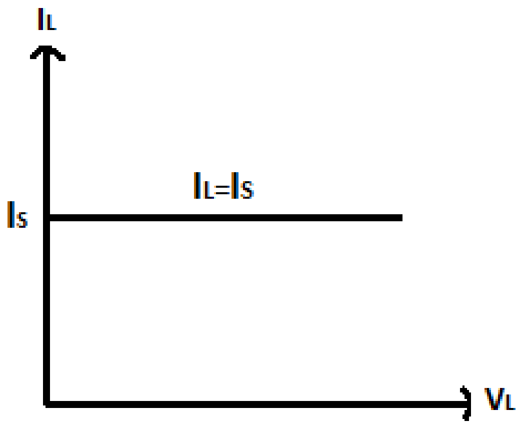
Characteristics



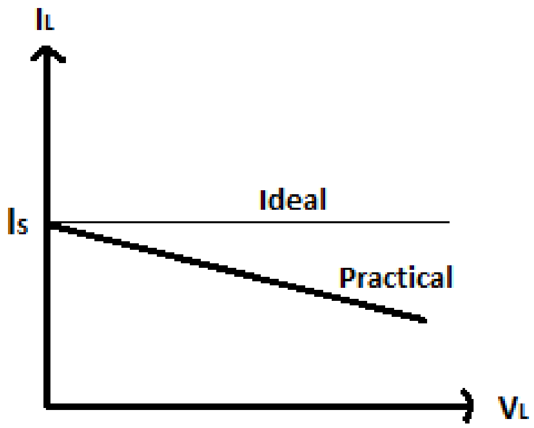
Ideal Current Source



Practical Current Source

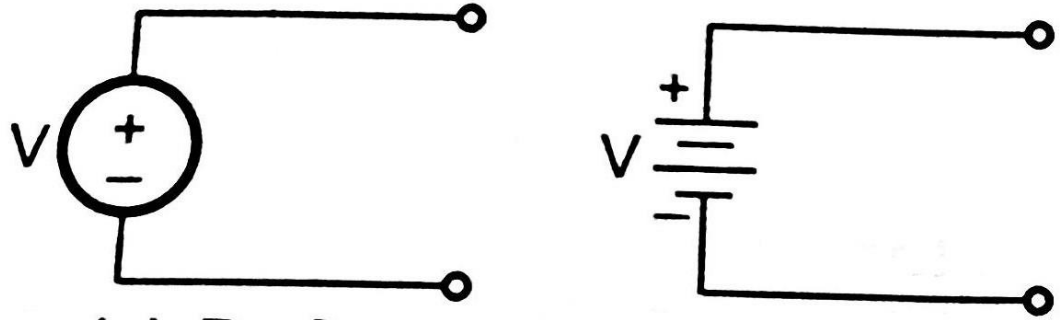


Characteristics

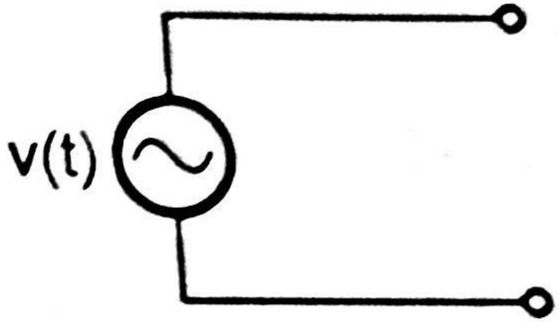


Characteristics

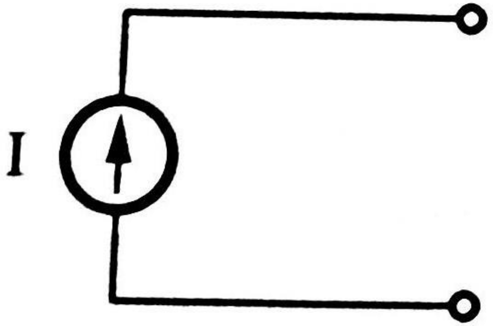
Classification of voltage & Current sources (Independent sources)



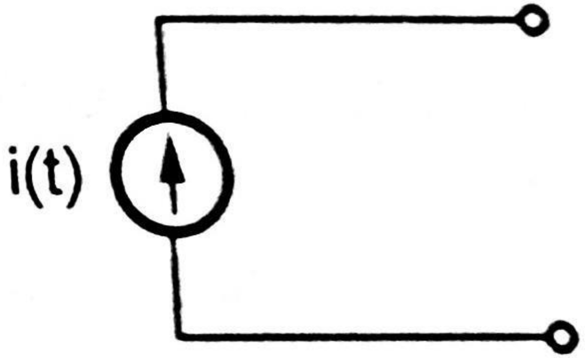
(a) D. C. voltage source



(b) A. C. voltage source

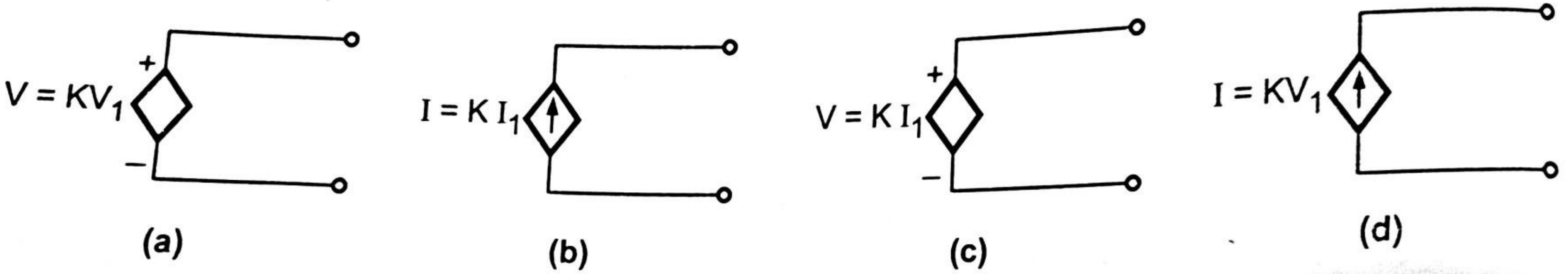


(c) D. C. current source



(d) A. C. current source

Dependent Sources



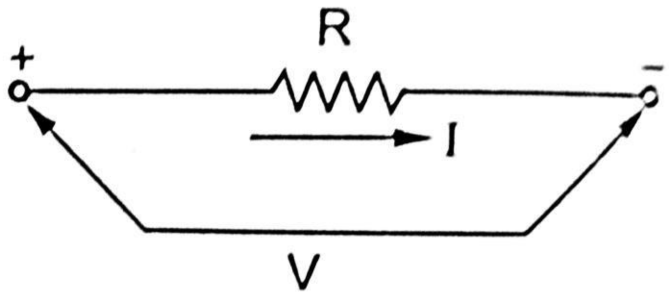
a. **Voltage Dependent Voltage source:** It produces the voltage as a function of voltage elsewhere in the given circuit.

b. **Current Dependent Current source:** It produces the current as a function of current elsewhere in the given circuit.

c. **Current Dependent Voltage source:** It produces the voltage as a function of current elsewhere in the given circuit.

d. **Voltage Dependent Current source:** It produces the current as a function of voltage elsewhere in the given circuit.

Ohm's Law

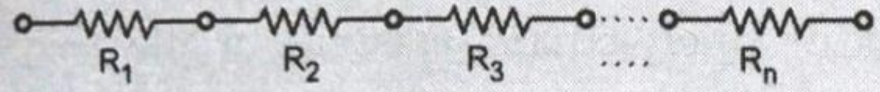
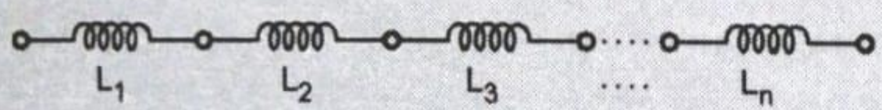
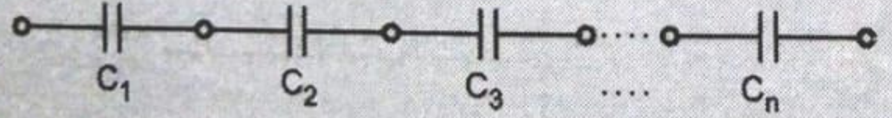


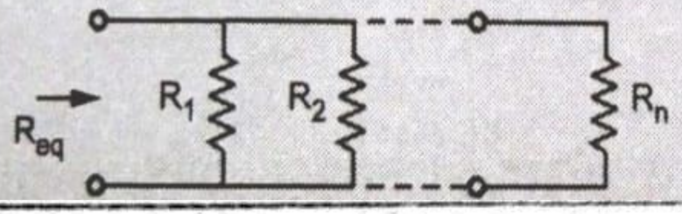
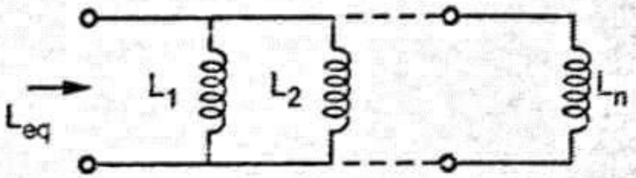
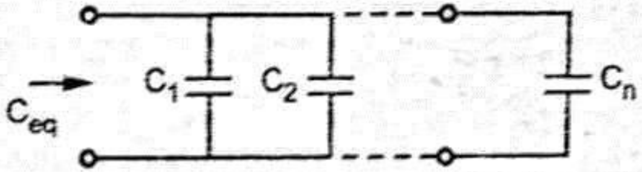
This Law gives the relationship between the potential difference (V), the current (I) and the resistance (R) of the DC circuit.

Statement: **The current flowing through electric circuit is directly proportional to the potential difference across the circuit and inversely proportional to the resistance of the circuit, provided the temperature remains constant.**

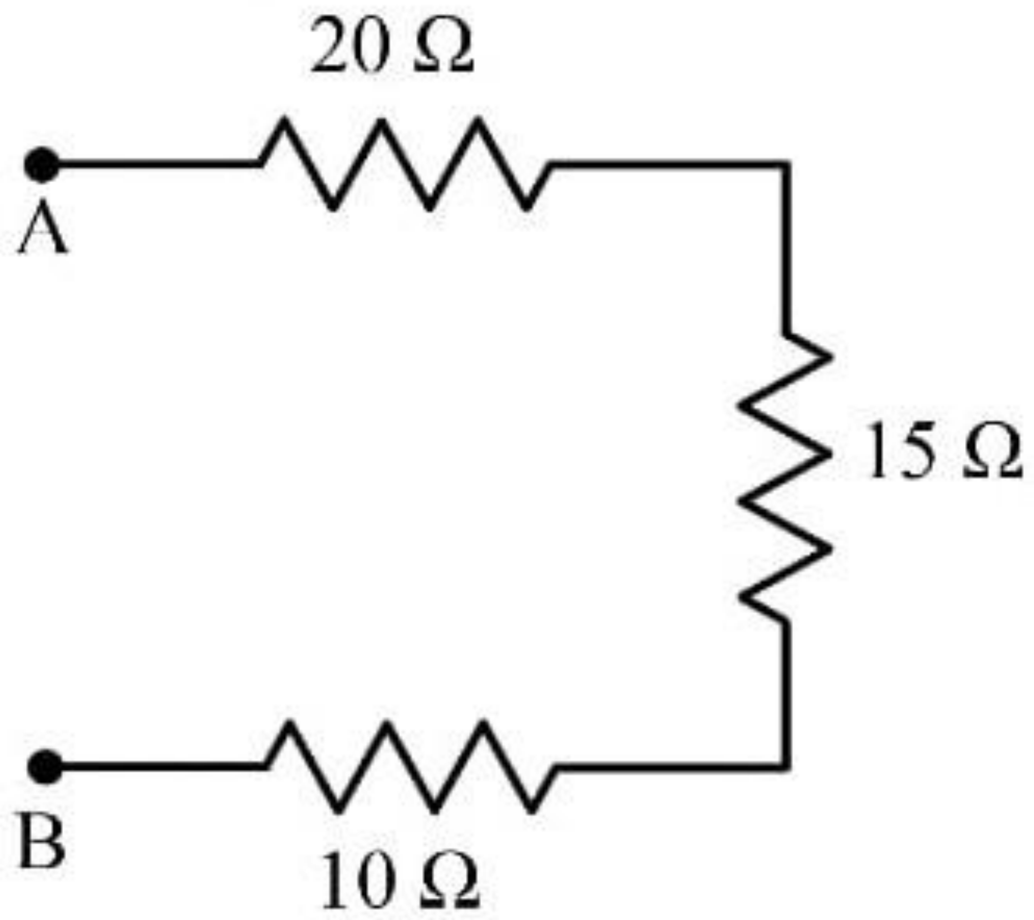
$$I \propto \frac{V}{R} \Rightarrow I = \frac{V}{R} \Rightarrow V = I\,R$$

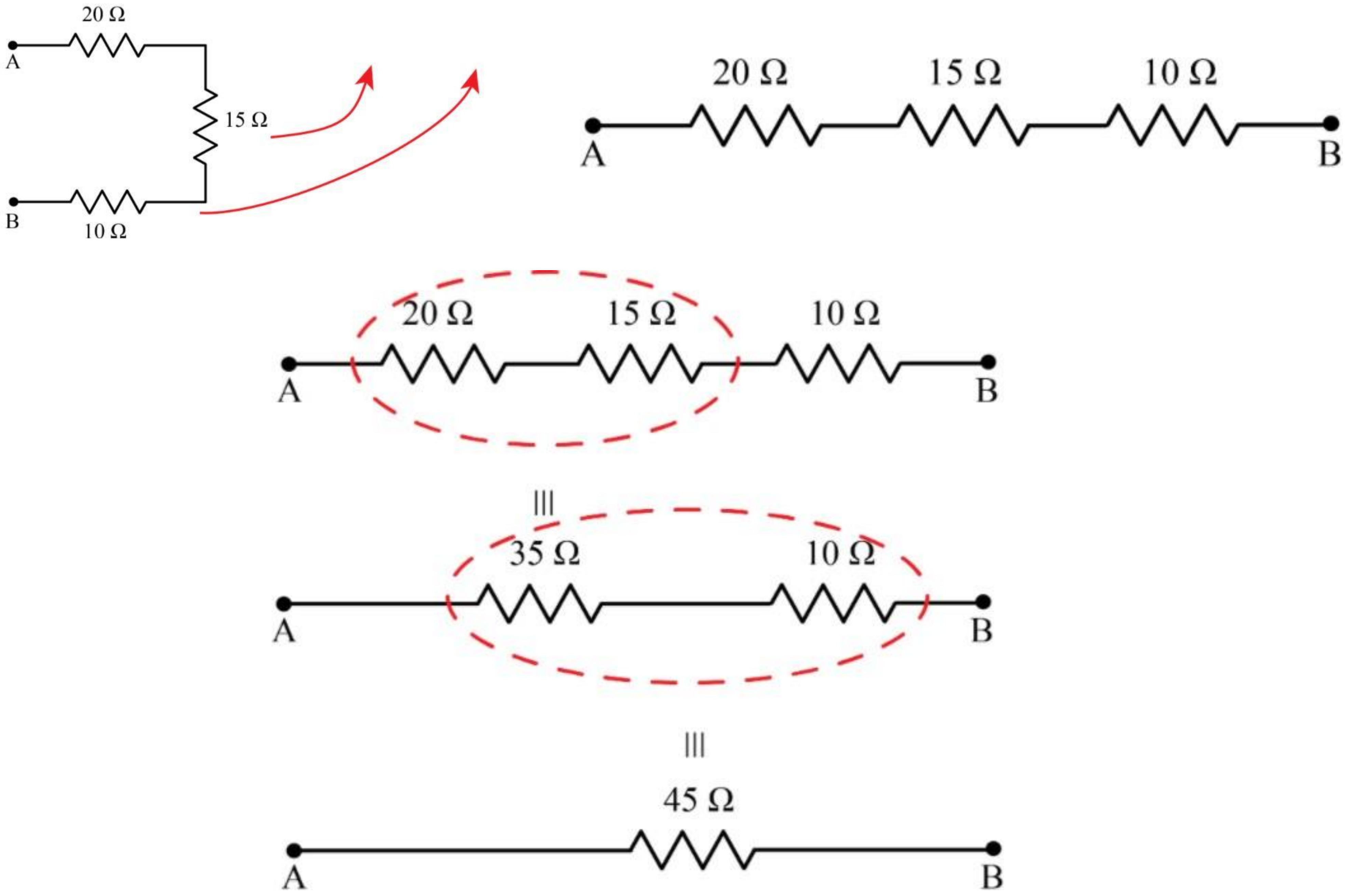
Series and Parallel Combination of Elements

Element	Equivalent
'n' Resistances in series  	$R_{eq} = R_1 + R_2 + R_3 + \cdots + R_n$
'n' Inductors in series  	$L_{eq} = L_1 + L_2 + \cdots + L_n$
'n' Capacitors in series  	$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}$

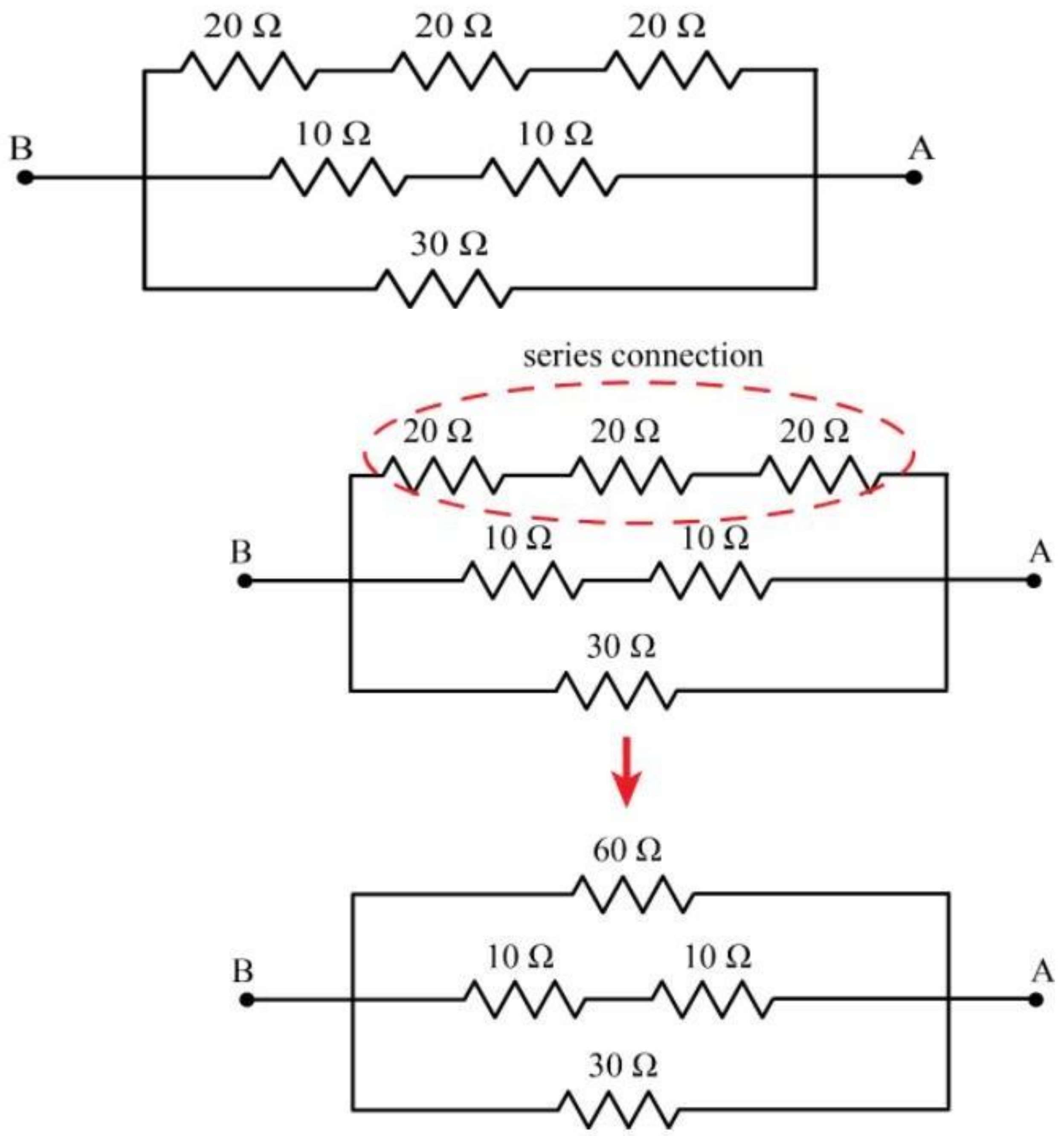
Element	Equivalent
'n' Resistances in parallel  	$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}$
'n' Inductors in parallel  	$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n}$
'n' Capacitors in parallel  	$C_{eq} = C_1 + C_2 + \cdots + C_n$

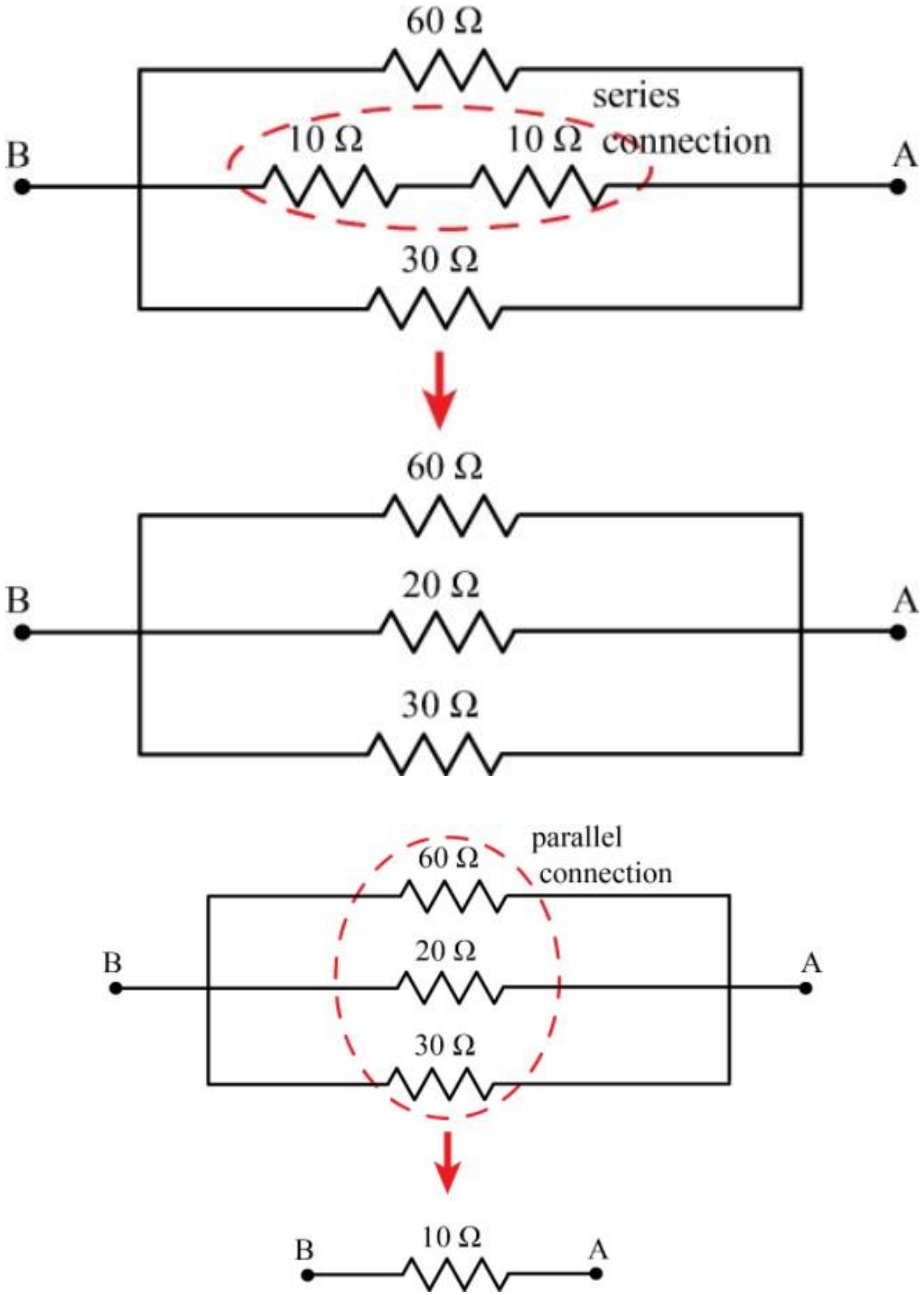
Find the equivalent resistance between points A and B.




20 Ω
A
15 Ω
B
10 Ω
20 Ω
15 Ω
10 Ω
A
B
20 Ω
15 Ω
10 Ω
A
B
35 Ω
10 Ω
A
B
45 Ω
A
B

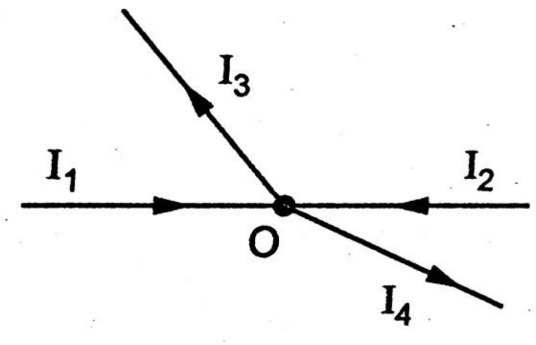
What is the equivalent resistance between points A and B?




60 Ω
series
connection
10 Ω
10 Ω
B
A
30 Ω
60 Ω
20 Ω
B
A
30 Ω
parallel
connection
60 Ω
20 Ω
B
A
30 Ω
10 Ω
B
A

Kirchhoff's Current Law

"The total Current flowing towards a junction point is equal to the total current flowing away from that junction point"



"The algebraic sum of all the current meeting at a junction point is always zero"

From the above figure,

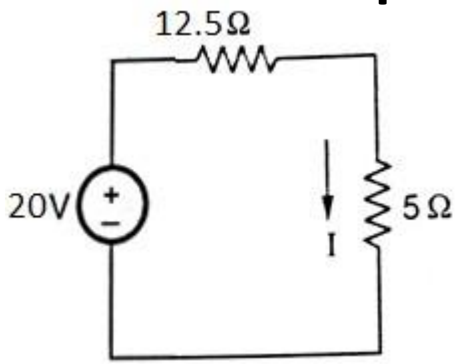
$$I_1 + I_2 = I_3 + I_4$$

(OR)

$$I_1 + I_2 + I_3 + I_4 = 0$$

Kirchhoff's Voltage Law

"In any closed loop network, the total voltage around the loop is equal to the sum of all voltage drops within the same loop"



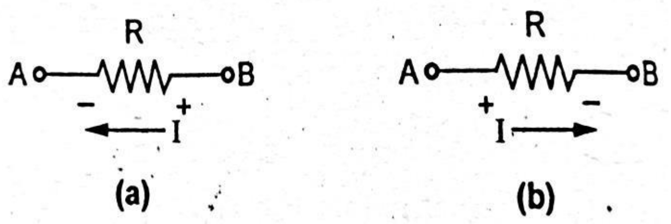
"The algebraic sum of all voltages within the loop must be equal to zero"

From the above figure,

$$20 = 12.5I + 5I$$

(OR)

$$20 - 12.5\,I - 5I = 0$$



When the current flows through resistance, the voltage drop occurs.

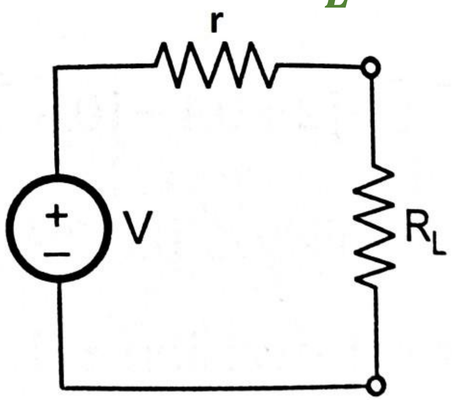
(a) I is flowing from right to left – point B potential > point A potential

(b) I is flowing from left to right – point A potential > point B potential

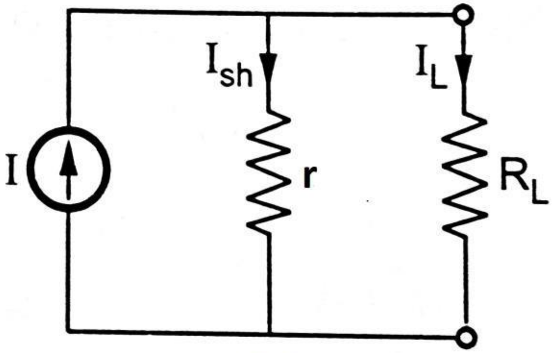
Source Transformation

While solving the electrical network, sometimes it is required to convert one type of source to another.

Two sources are said to be identical, when they produce identical terminal voltage V_L & load current I_L.



(a)



(b)

The above figures represent a practical current & voltage source with load connected to both the sources.

From fig(a)

$$I_L = \frac{V}{r+R_L} \text{-------------(1)}$$

From fig(b)

$$I_L = \frac{I \, x \, r}{r+R_L} \text{--------------(2)}$$

From fig(a)

$$I_L = \frac{V}{r+R_L} \text{ -------------(1)}$$

From fig(b)

$$I_L = \frac{I \ x \ r}{r+R_L} \text{ --------------(2)}$$

From (1) and (2),

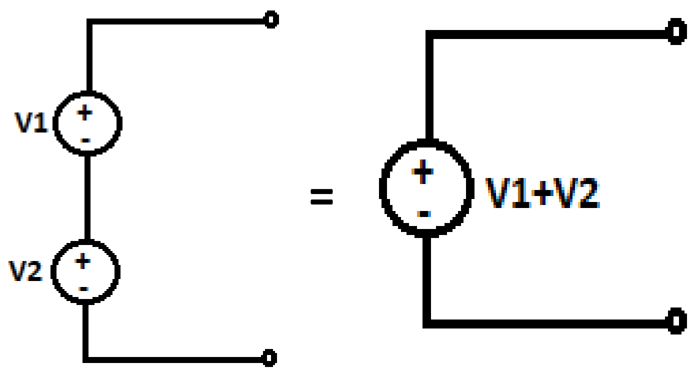
$$\frac{V}{r+R_L} = \frac{I \ x \ r}{r+R_L}$$

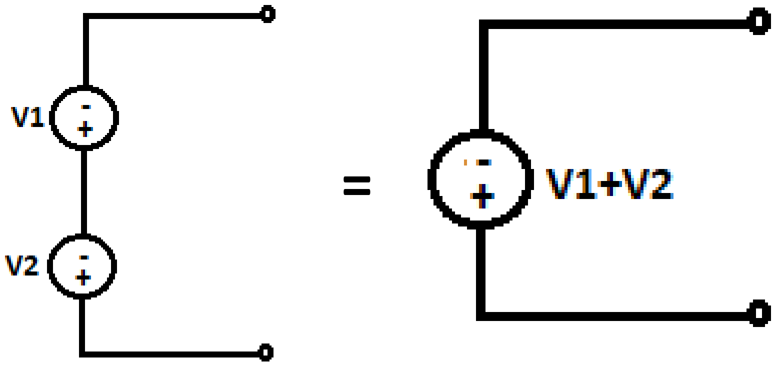
$\Rightarrow$ V= I r (or) $I = \frac{V}{r}$

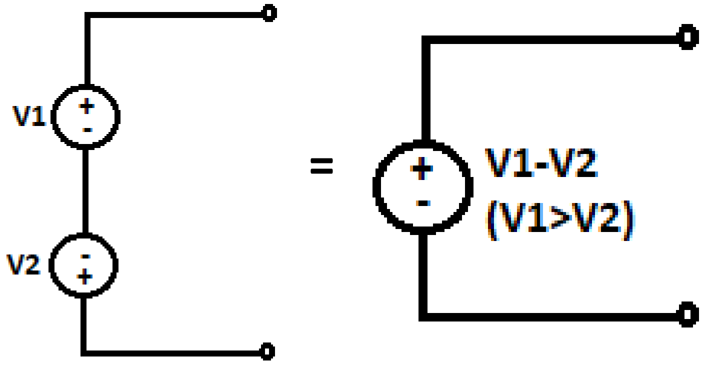
Hence, A voltage source V in series with its internal resistance r can be converted into a current source I = $\frac{V}{r}$, with same 'r' connected in parallel with it.

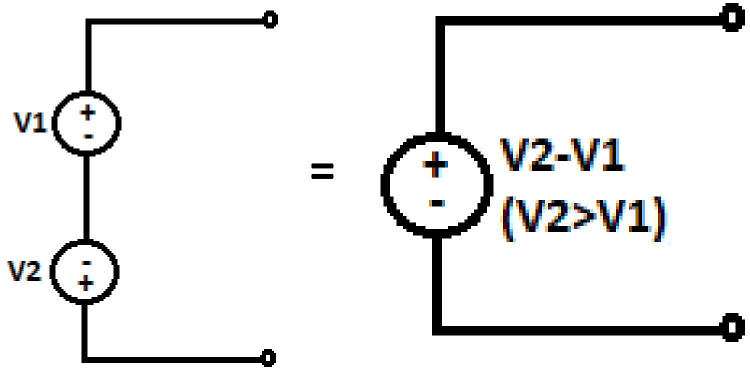
Combination of Sources

1. Voltage Sources in Series



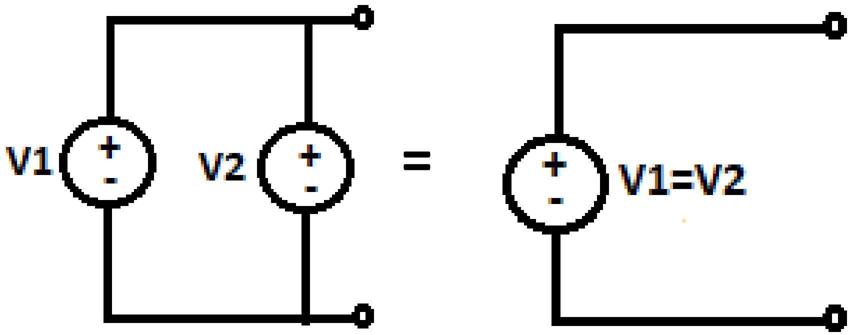




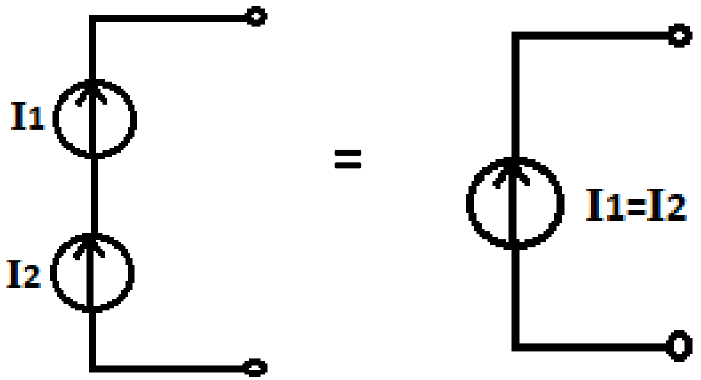


Combination of Sources

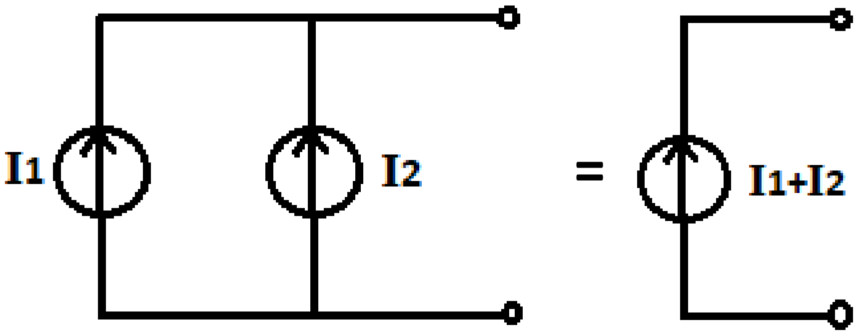
2. Voltage Sources in Parallel

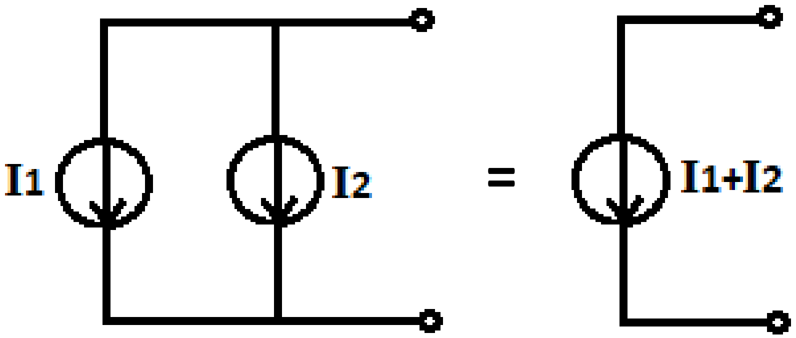


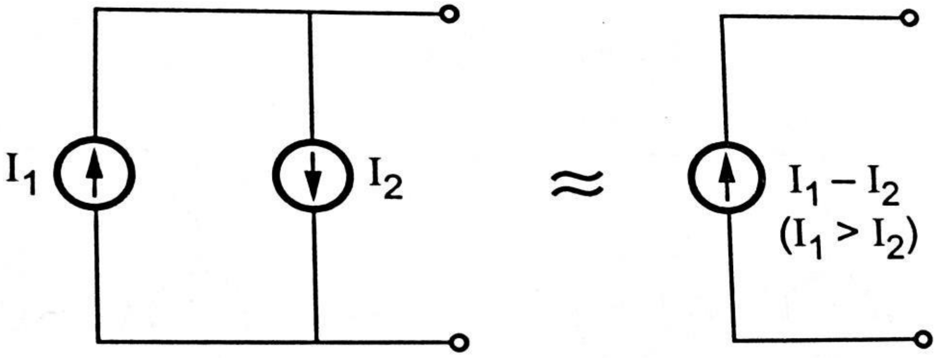
3. Current Sources in Series

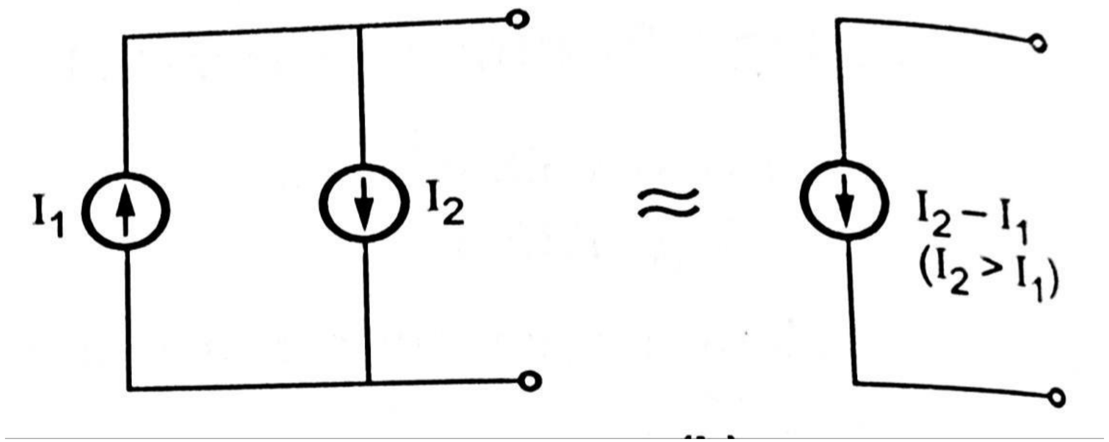


4. Current Sources in Parallel

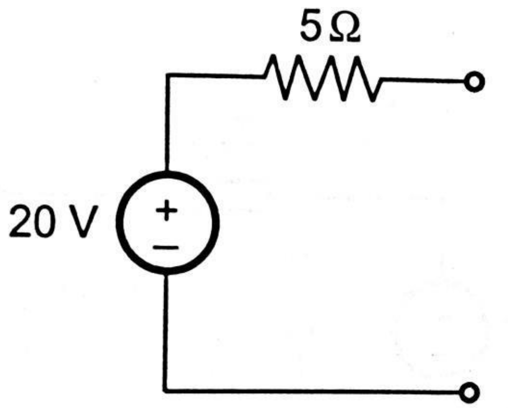




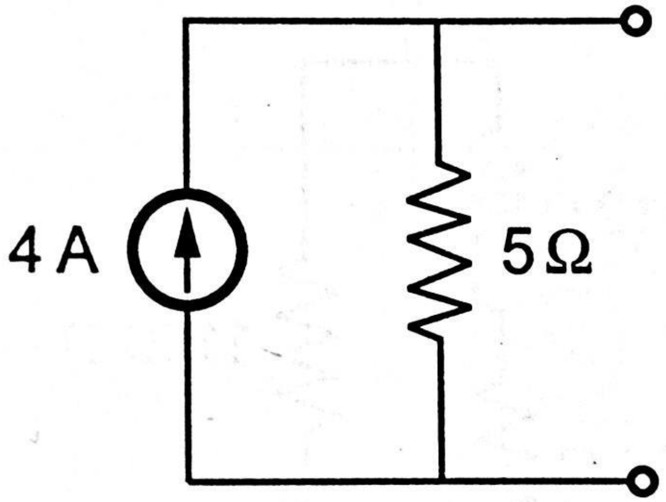




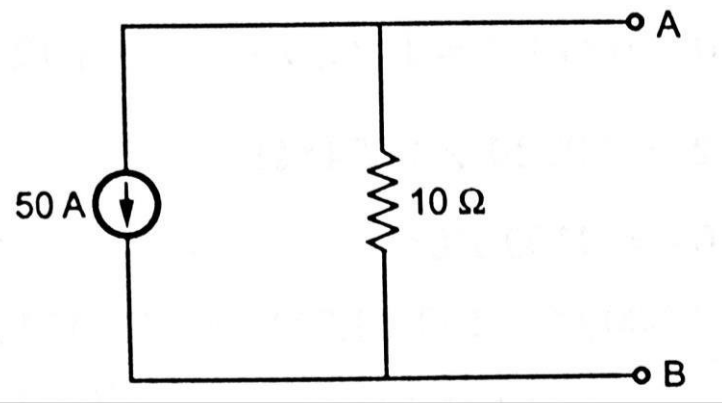
Problem1: Transform a voltage source of 20 volts with an internal resistance of 5Ω to a current source.



Then the current of current source is, $\mathbf{I = \frac{V}{r} = \frac{20}{5} = 4A}$ with internal parallel resistance same as ‘r’.



Problem2: Convert the given current source of 50 A with internal resistance of 10 Ω to the equivalent voltage source.

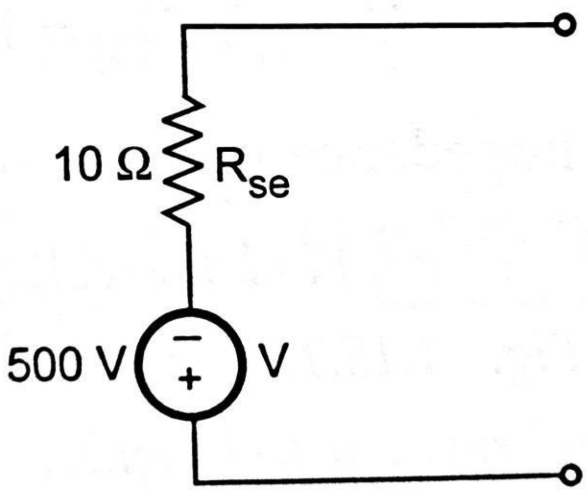


The given values are I=50 A and r = 10 Ω.

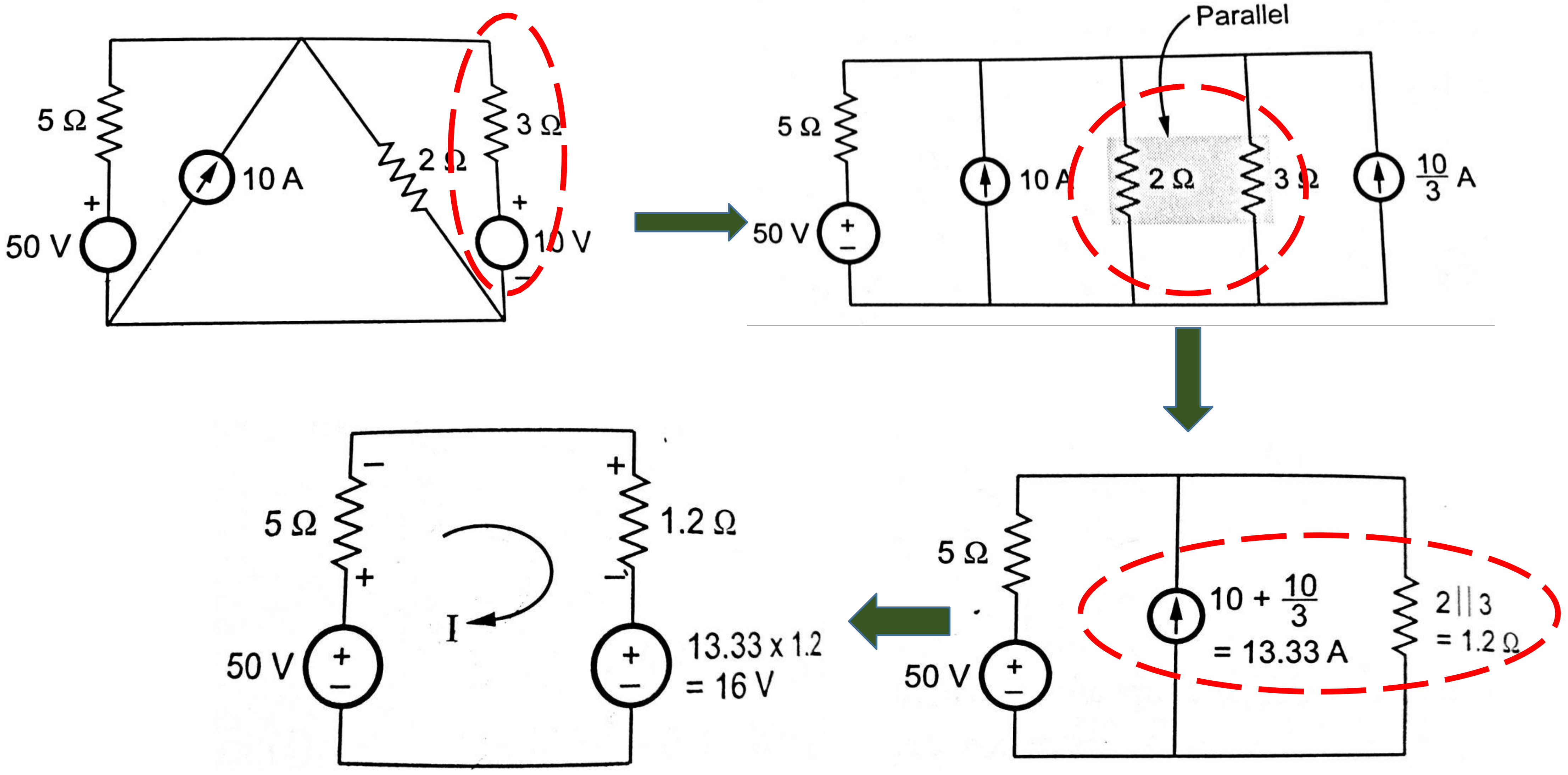
V = I x r = 50 x 10

= 500 V

R_{se}= R_{sh} = r = **10 Ω** in series



Problem 3: Using source transformation find power delivered by 50 V source in given network.



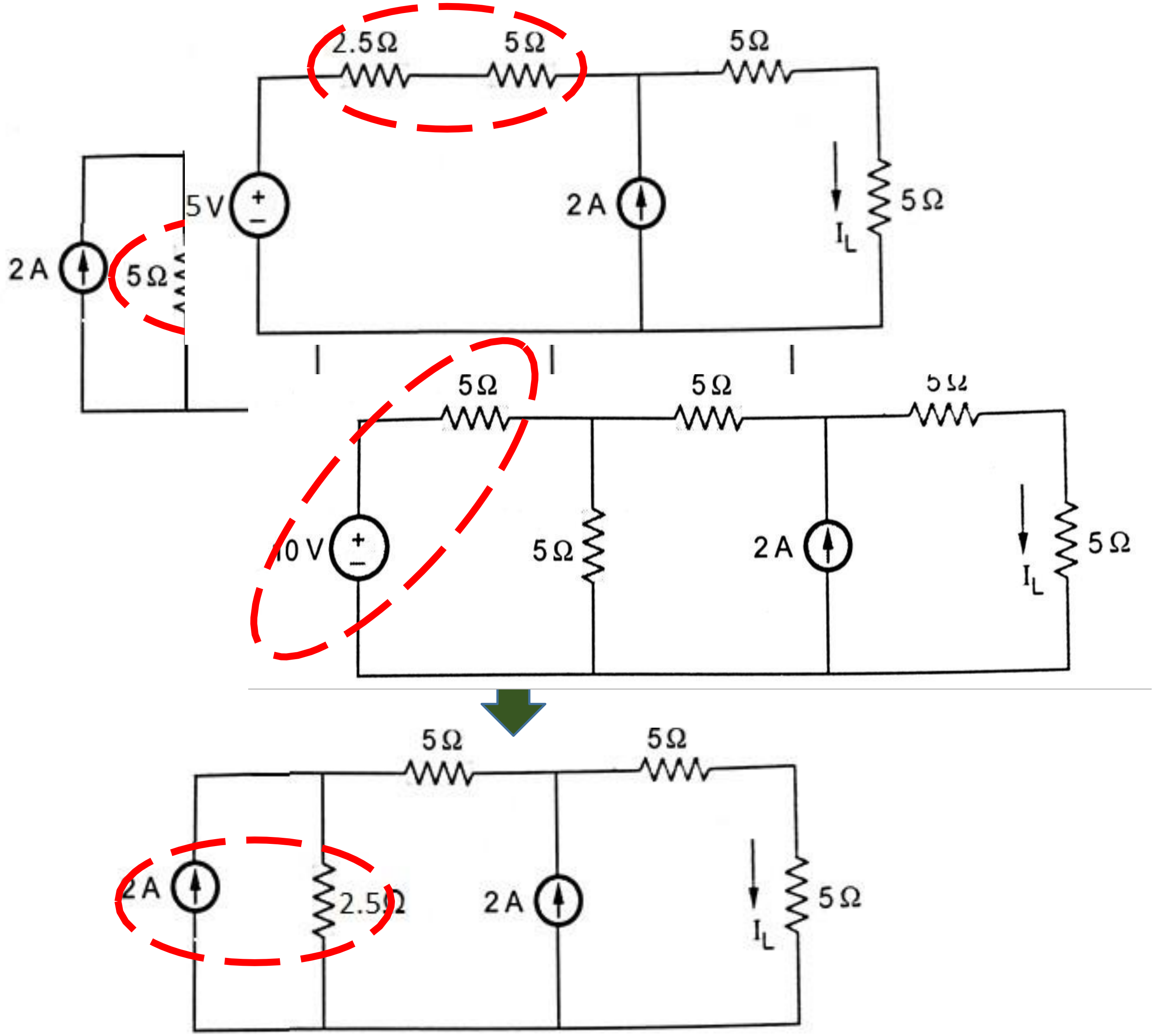
Applying KVL, 50-5I-1.2I-16=0

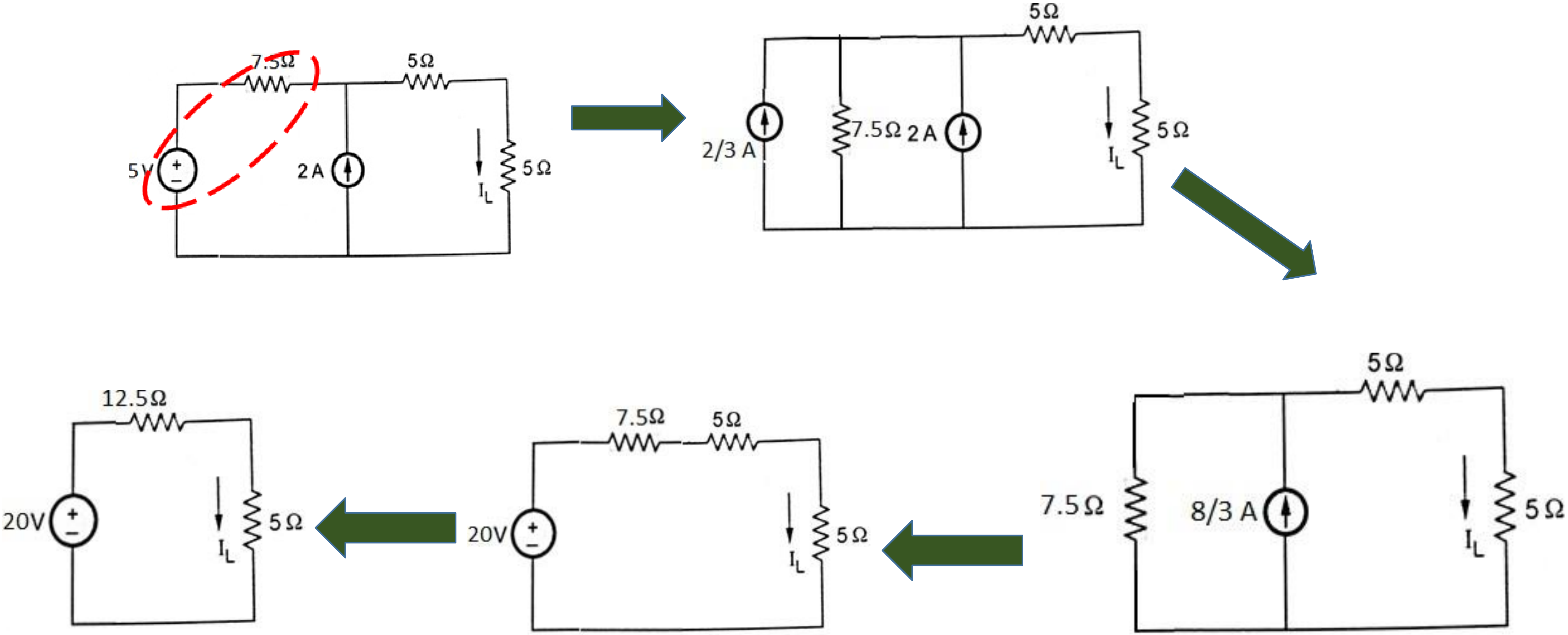
I = 5.484 A

Power delivered by 50V source = V I

= 50 x 5.484 = 274.2W

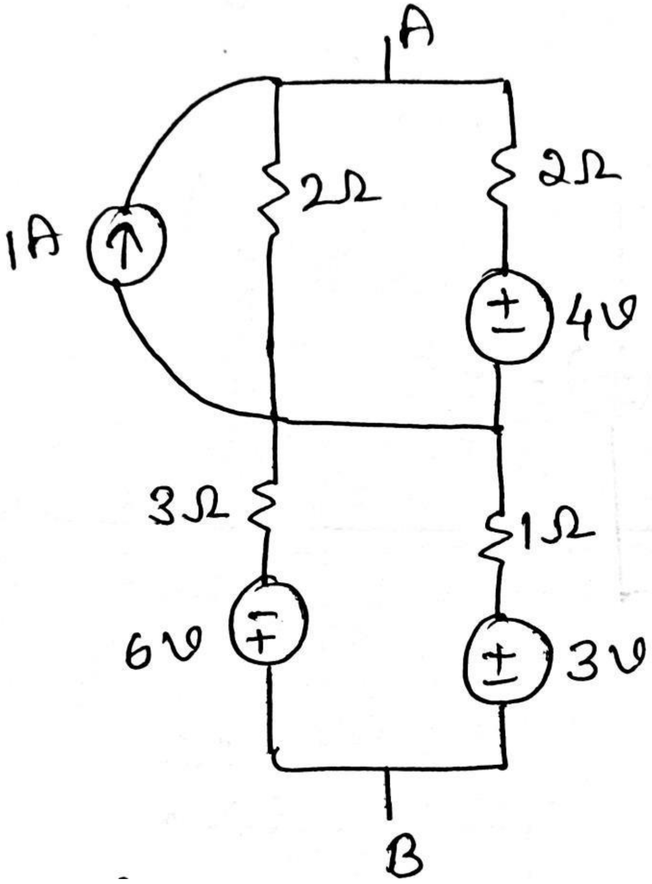
Problem 4: Using Source transformation find the load I_L in the following circuit.



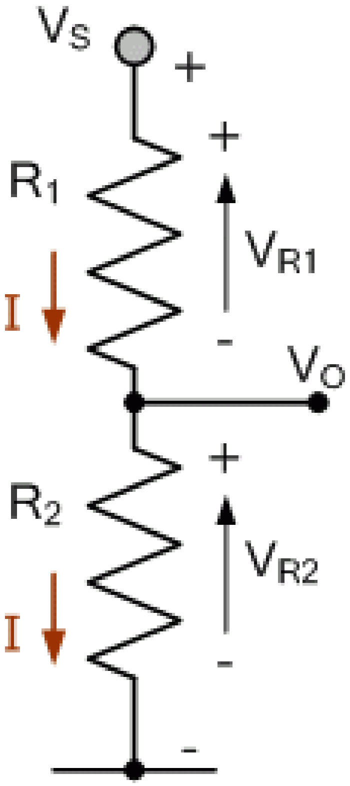


$$I_L = \frac{20}{12.5+5} = 1.1428\ \text{A}$$

Problem 5: **Simplify the network into a single current source.**



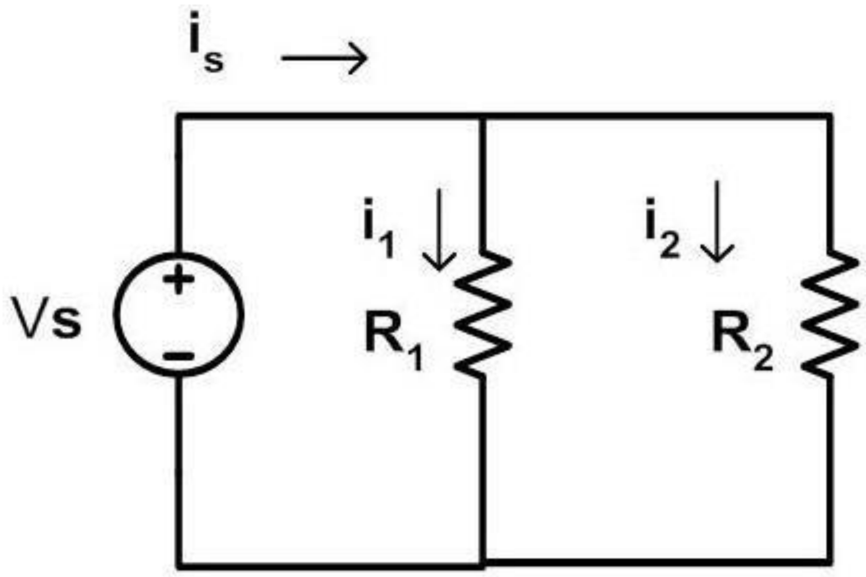
Voltage Division Rule



$$V_{R1} = \frac{V_S \,.\, R_1}{R_1 + R_2}$$

$$V_{R2} = \frac{V_S \,.\, R_2}{R_1 + R_2}$$

Current Division Rule



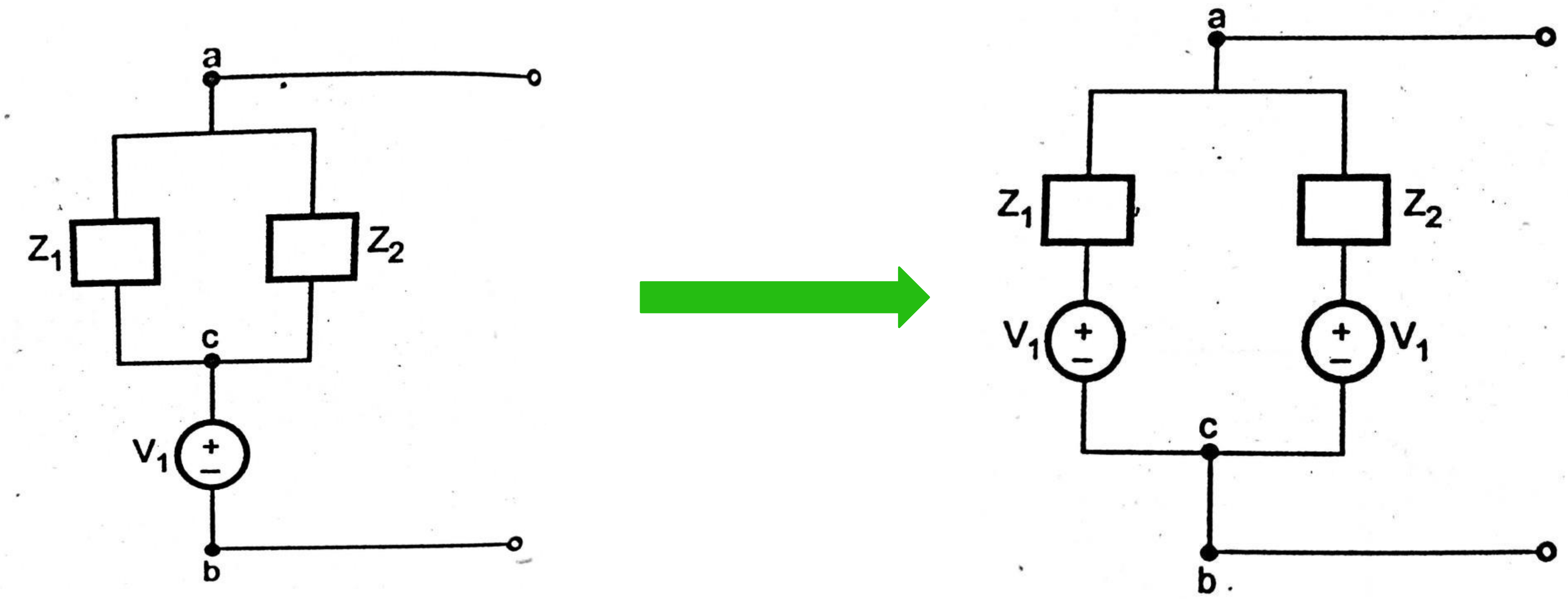
$$I_1 = \frac{I_S \,.\, R_2}{R_1 + R_2}$$

$$I_2 = \frac{I_S \,.\, R_1}{R_1 + R_2}$$

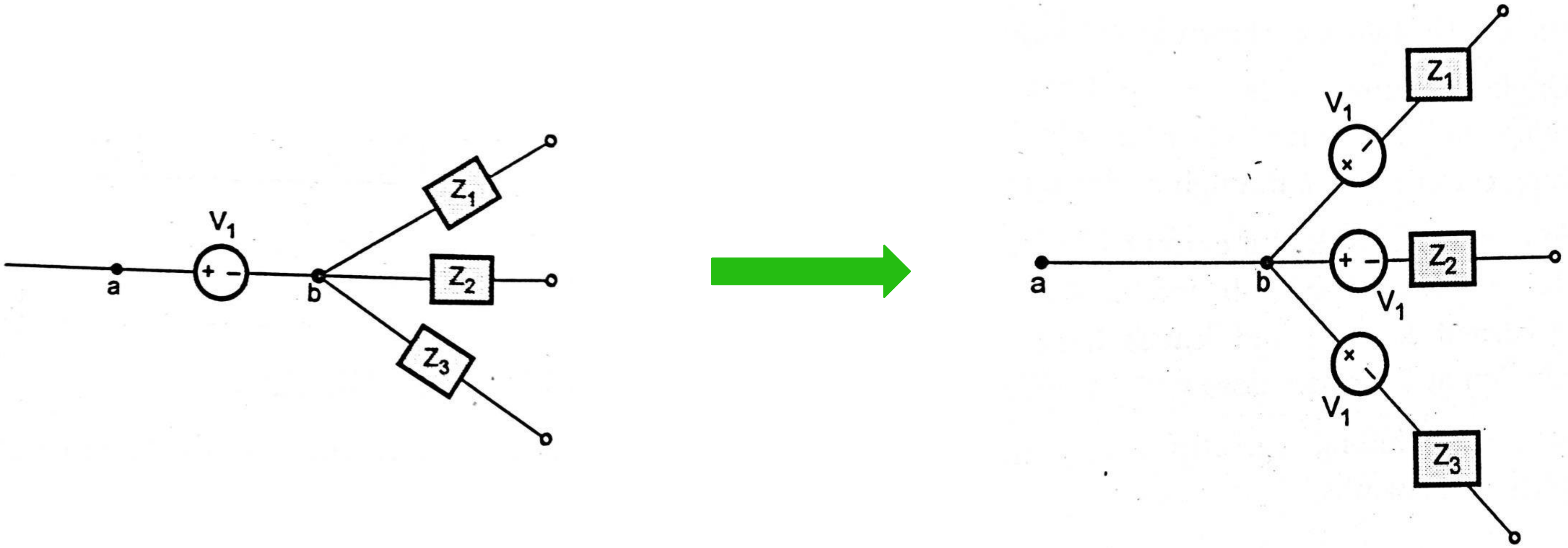
Source Shifting

In a Network, if there is no impedance in series with voltage source and there is no impedance in parallel with a current source, then source transformation can not be applied.

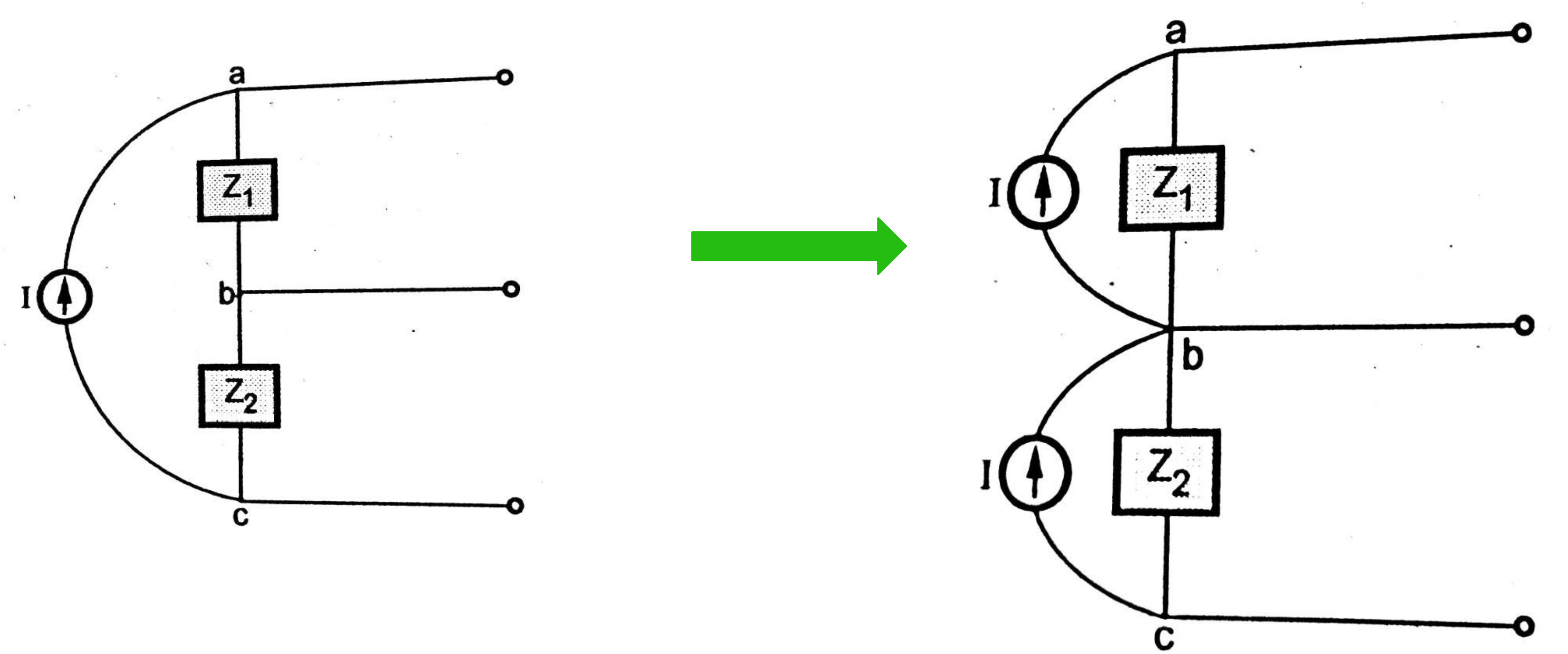
1. Voltage Source Shifting



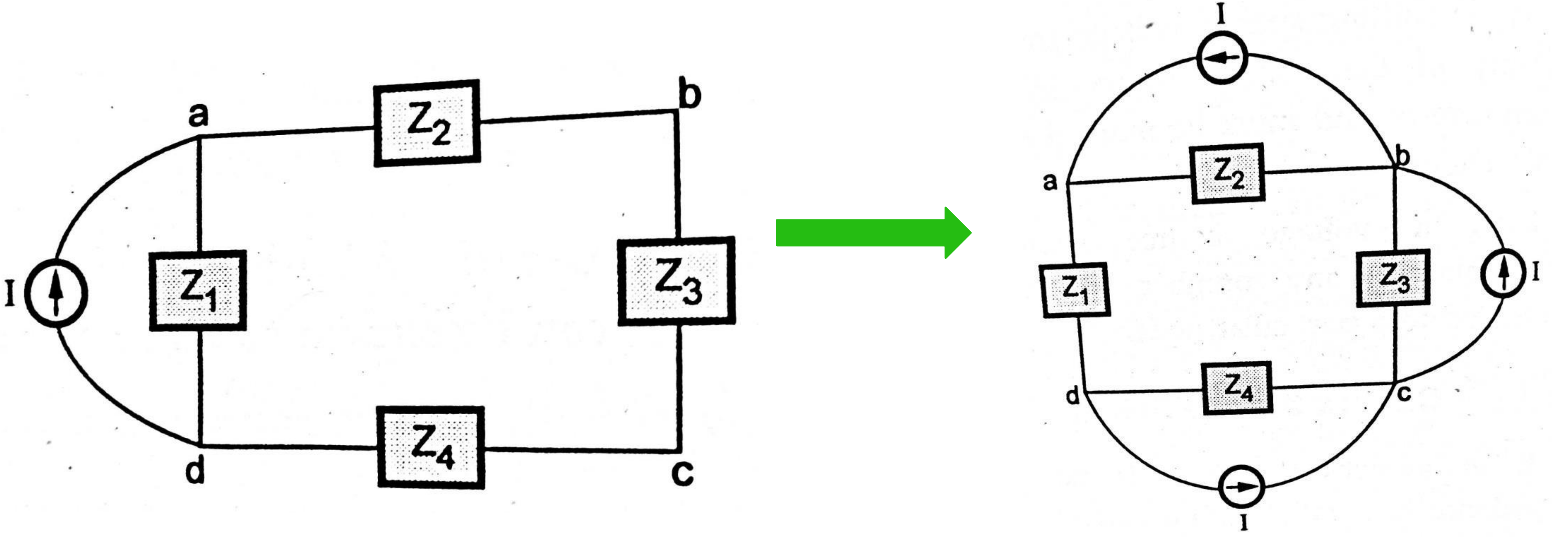
1. Voltage Source Shifting



2. Current Source Shifting

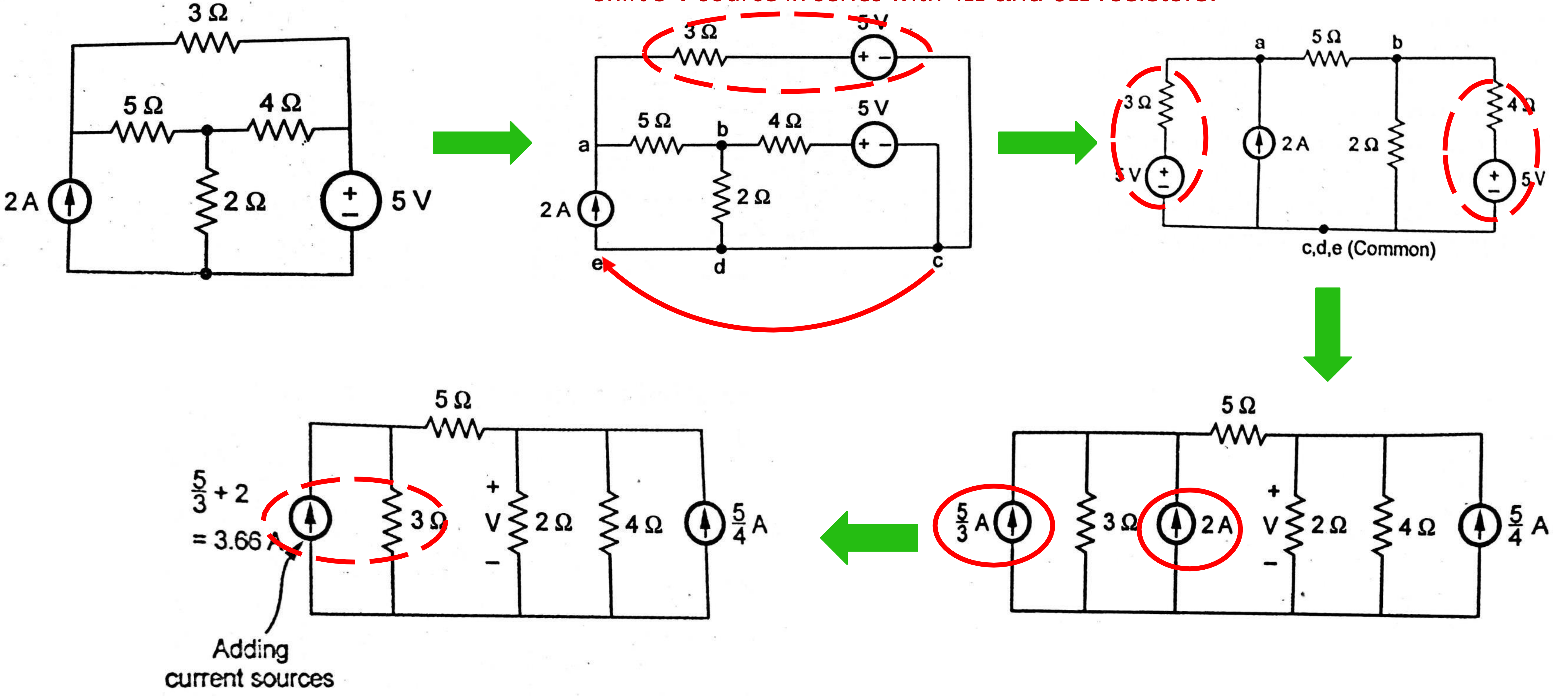


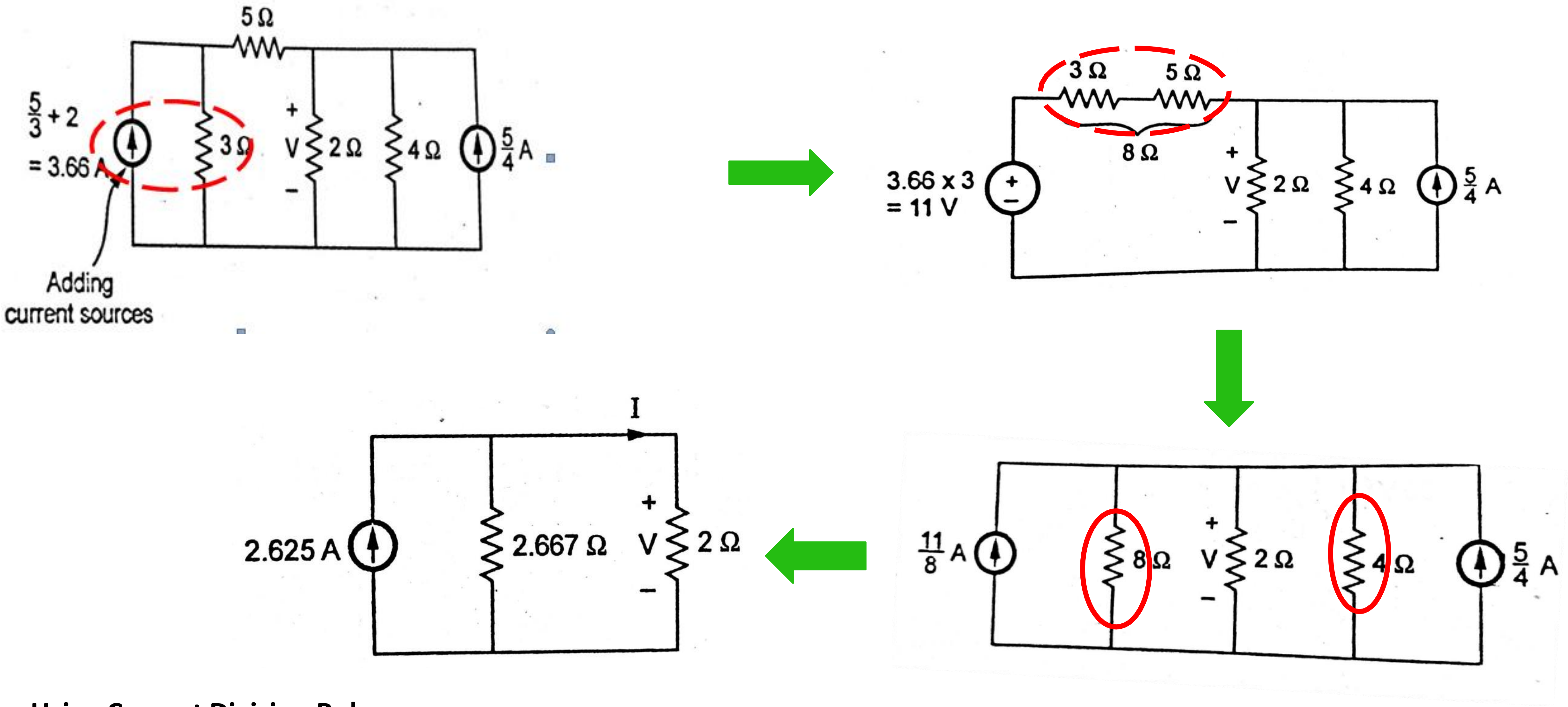
2. Current Source Shifting



Problem 1: Using source transformation and source shifting techniques, find voltage across 2Ω resistor.

Shift 5 V source in series with 4Ω and 3Ω resistors.



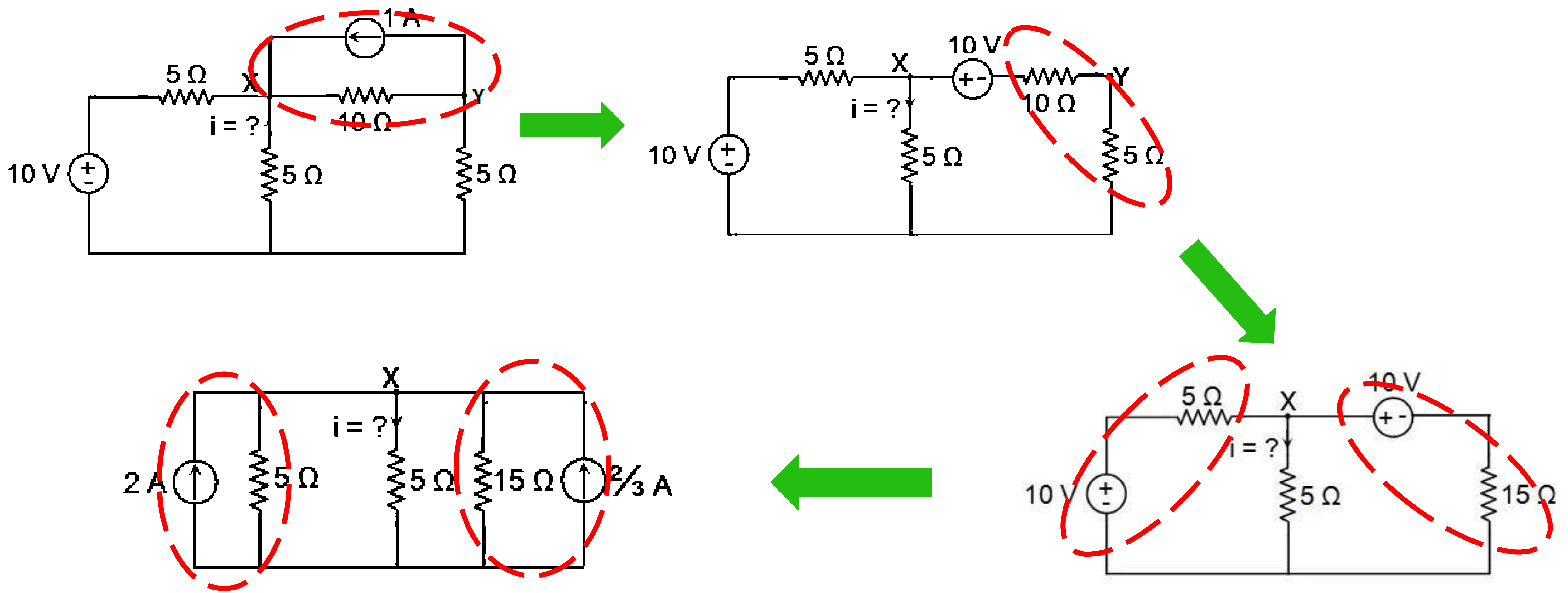


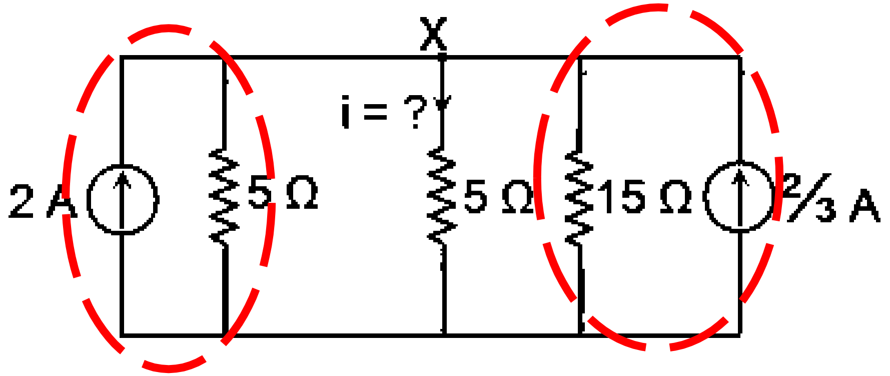
Using Current Division Rule,

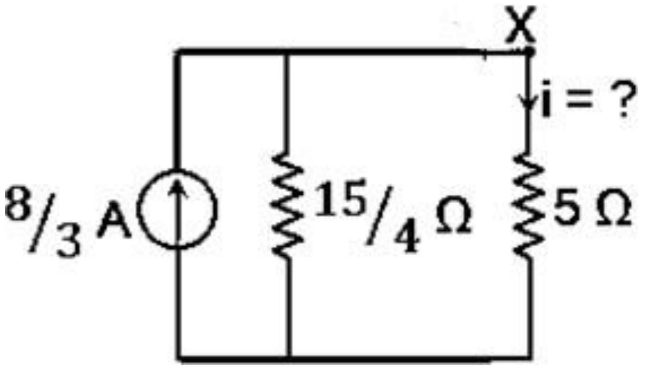
$$I = 2.625 \text{ x } \frac{2.667}{2.667+2} = 1.5 \text{ A}$$

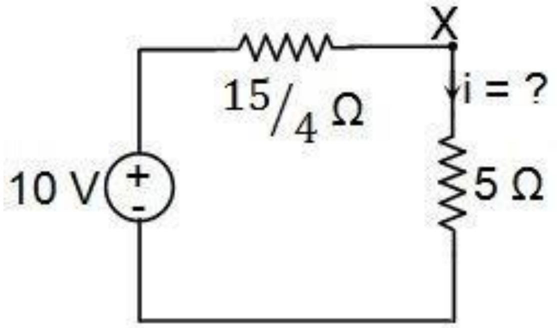
$$V = I \text{ x } 2 = 1.5 \text{ x } 2 = 3 \text{ V}$$

Problem 2: **Reduce the network shown in figure and find 'i' using source shifting and source transformation.**









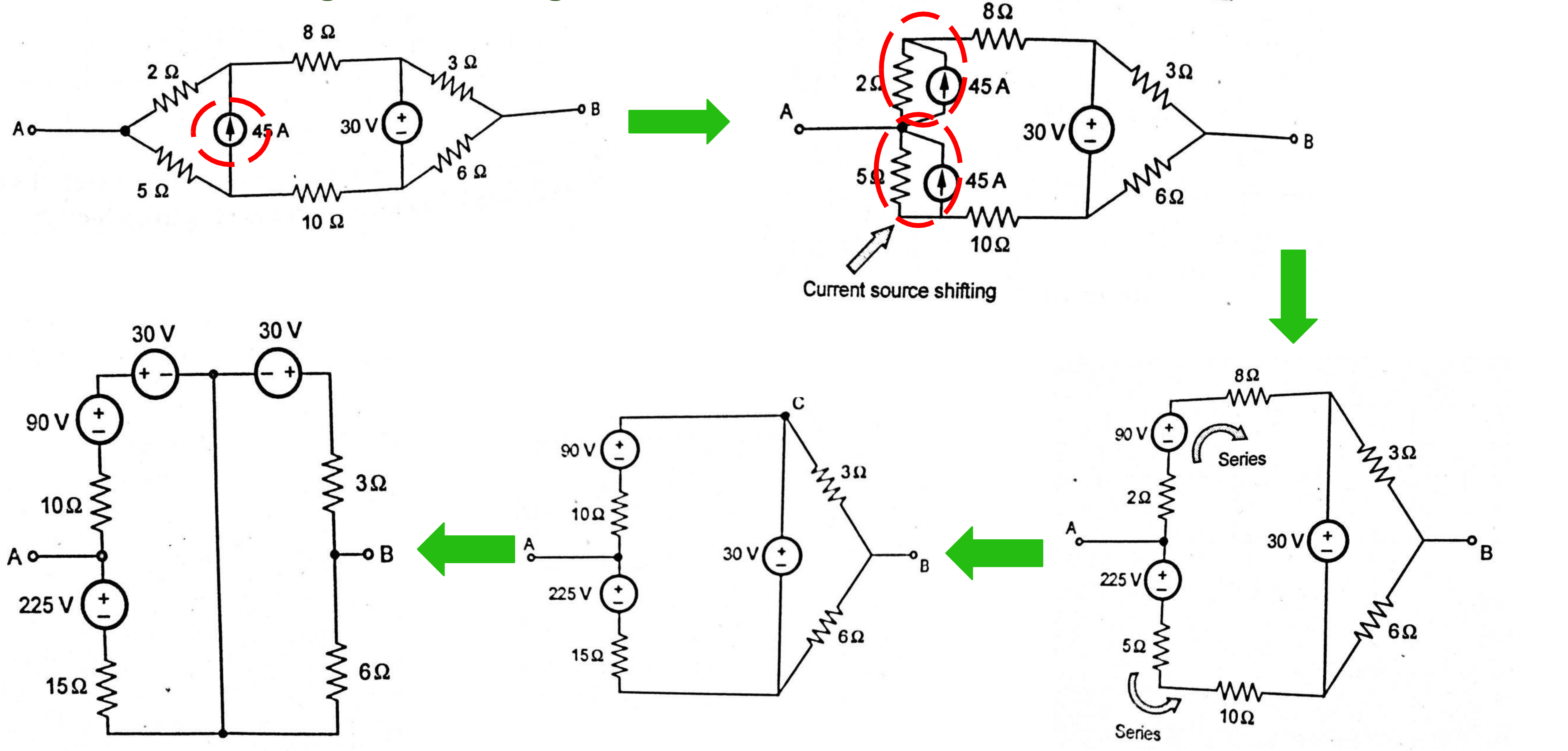
Applying KVL,

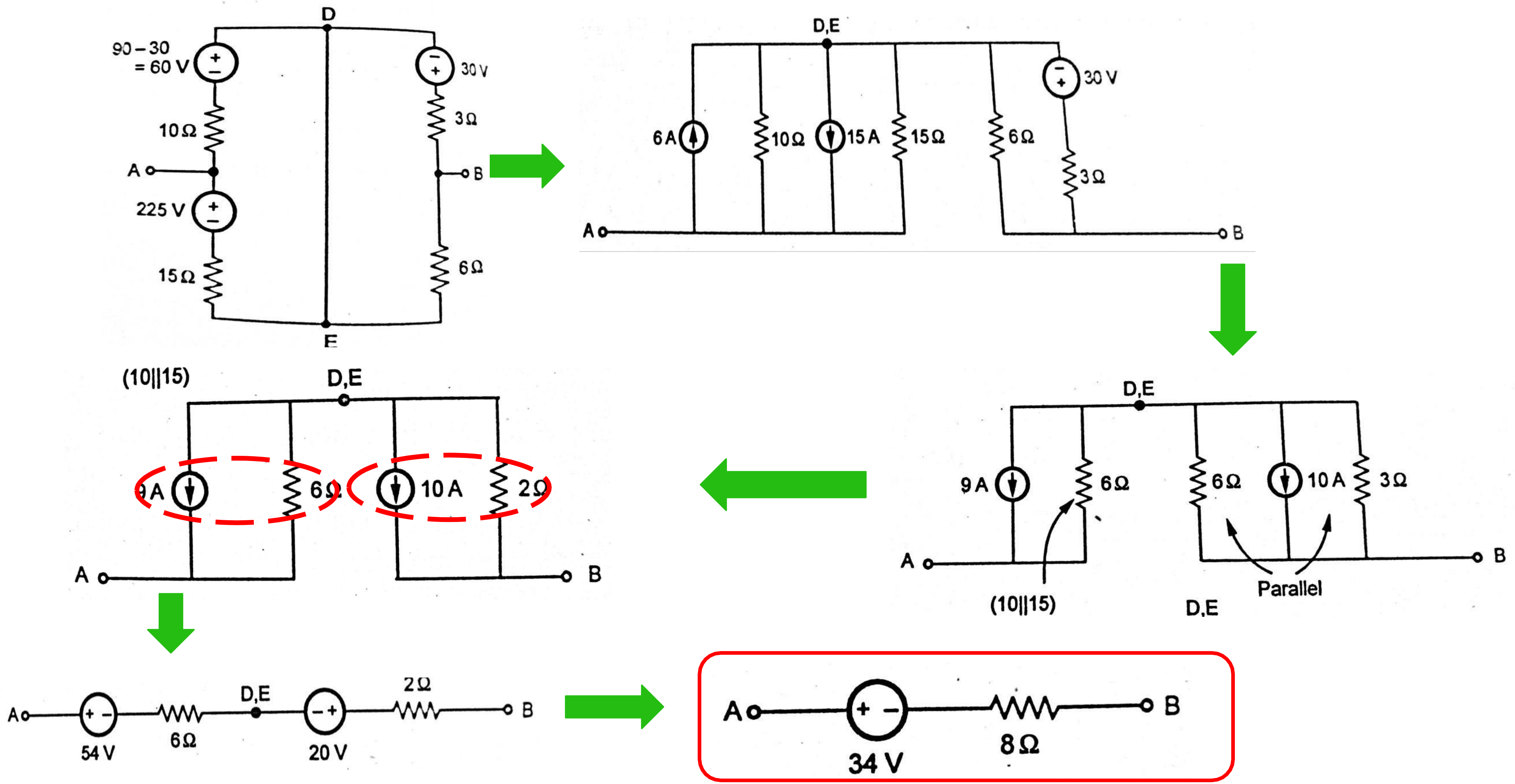
$$10 - 15/4\ i - 5i = 0$$

$$10 = (15/4+5)\ i$$

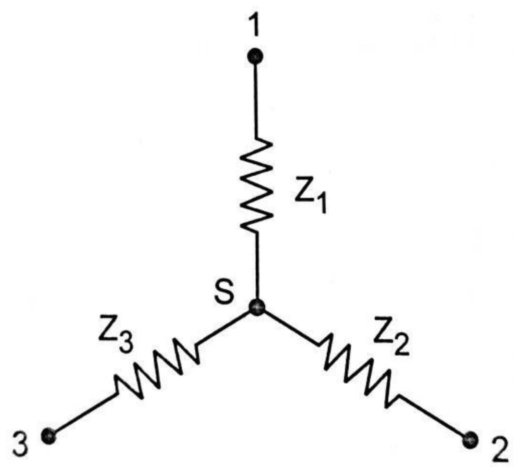
$$i = 8/7\ A = 1.1428\ A$$

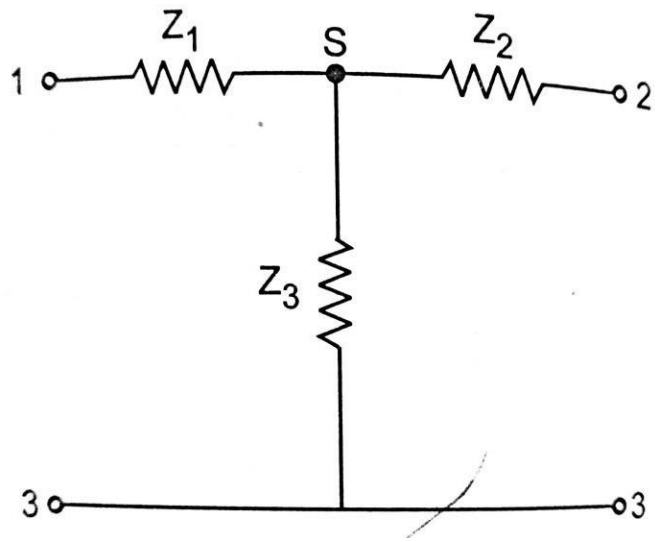
Problem 3: Reduce the network shown in figure to a single voltage source in series with a resistance using source shifting and source transformation.



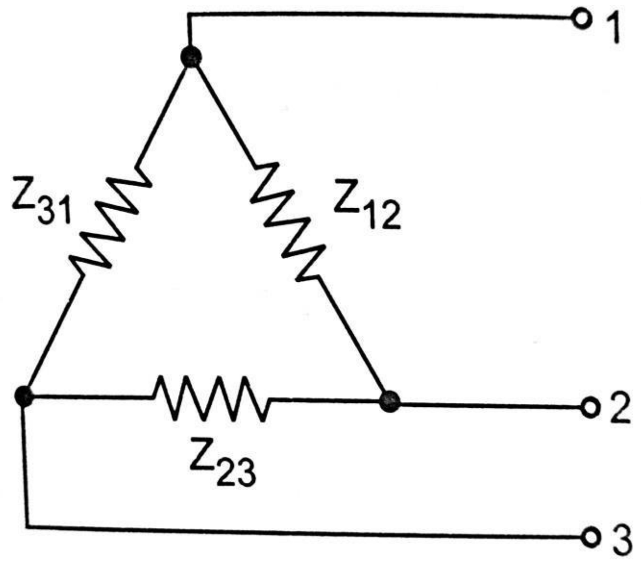

D
90 – 30
= 60 V
30 V
10 Ω
3 Ω
A
B
225 V
6 Ω
15 Ω
E
D,E
6 A
10 Ω
15 A
15 Ω
6 Ω
30 V
3 Ω
A
B
(10||15)
D,E
9 A
6 Ω
10 A
2 Ω
A
B
D,E
9 A
6 Ω
6 Ω
10 A
3 Ω
A
B
(10||15)
D,E
Parallel
A
54 V
6 Ω
D,E
20 V
2 Ω
B
A
34 V
8 Ω
B

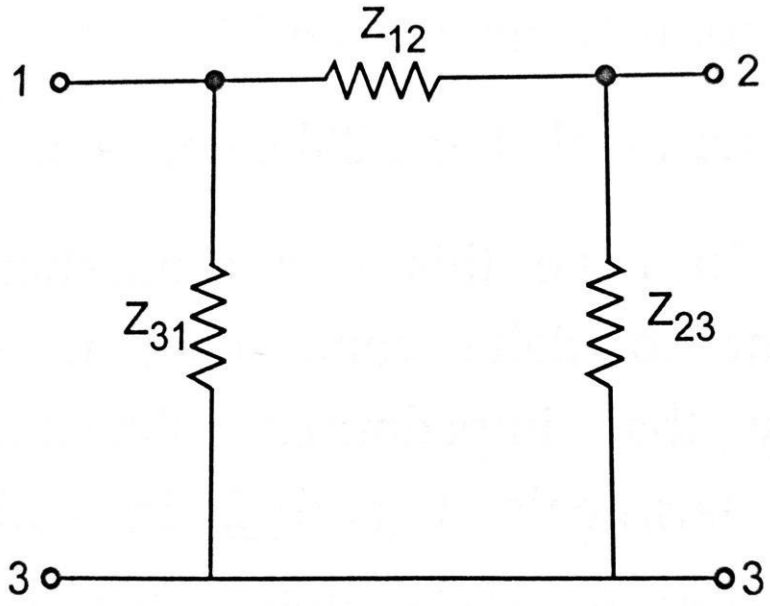
Star-Delta & Delta-Star Transformation



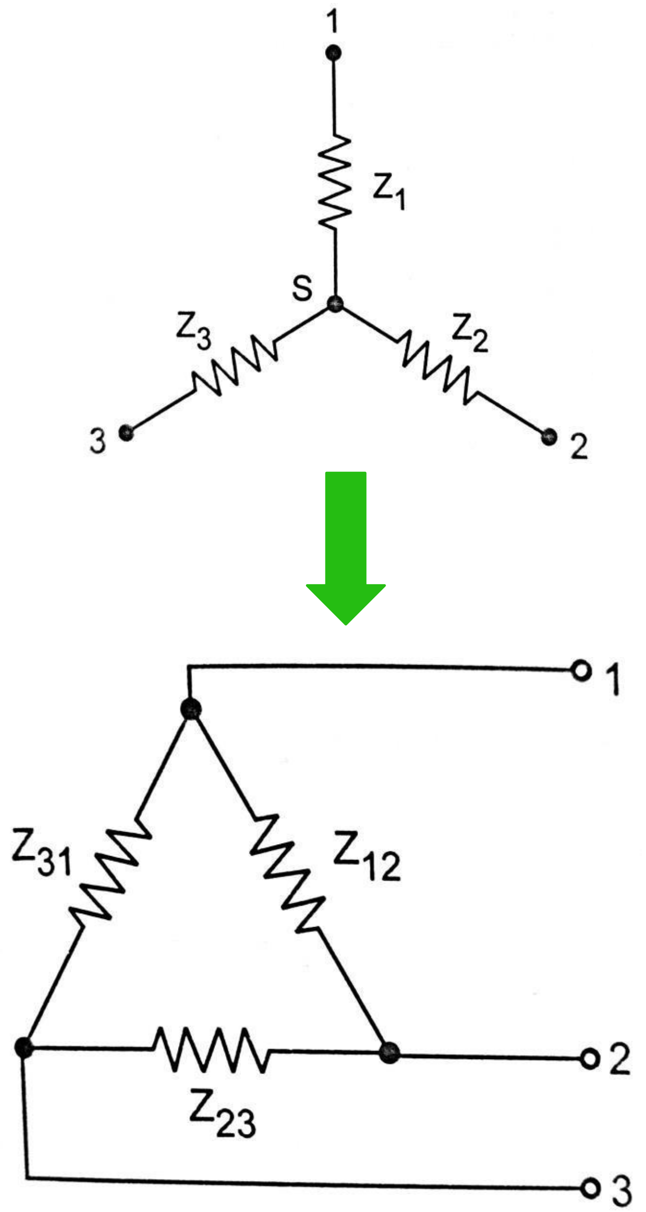


If 3 impedances are connected in such a manner that one end of each is connected together to form a junction point called "star point" and impedances are said to be connected in Star fashion (T connection).

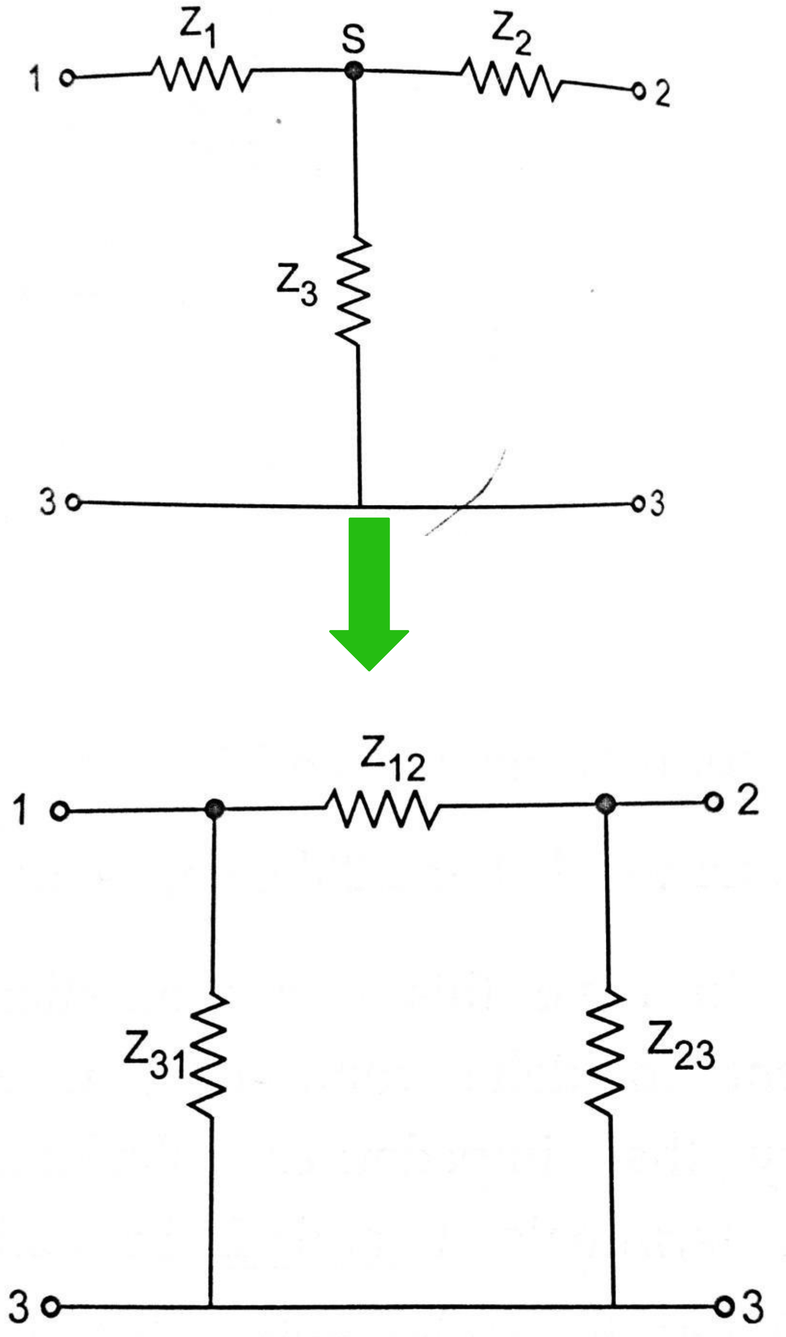




If 3 impedances are connected in such a manner that one end of the first is connected to first end of the second, the second end of the second is connected to first end of third and so on to complete a loop then the impedances are said to be connected in “Delta” and the delta connection (Pi Connection) is always a closed path.



1
Z_1
S
Z_3
Z_2
3
2
1
Z_{31}
Z_{12}
Z_{23}
2
3



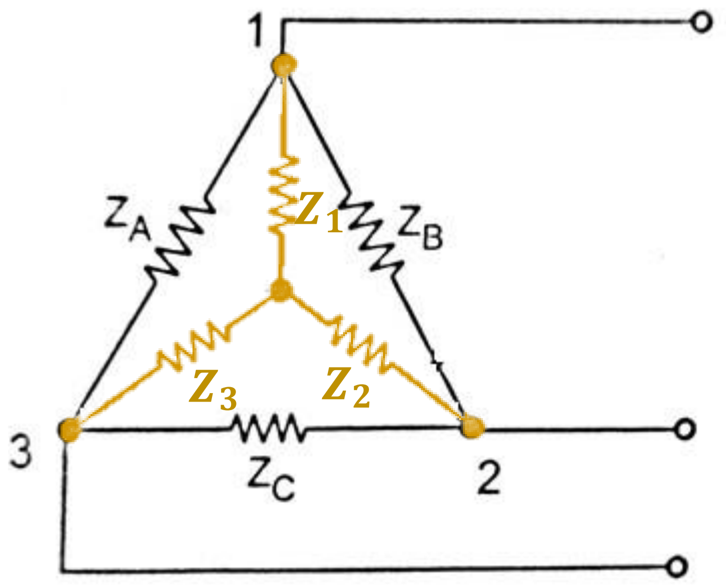
Z_1
S
Z_2
1
2
Z_3
3
3
Z_{12}
1
2
Z_{31}
Z_{23}
3
3

Delta-Star / π to T Transformation

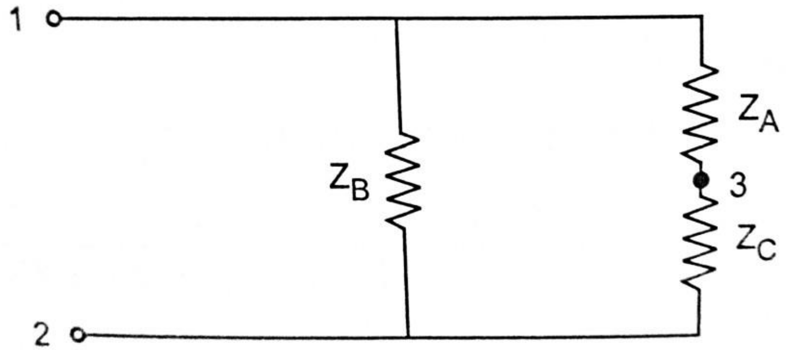
Consider 3 impedances Z_A, Z_B & Z_C connected in Delta as shown.

The terminals between which these are connected in delta are named as 1,2,3.

It is always possible to replace these delta by 3 equivalent star connected impedances Z_1, Z_2 & Z_3 between the same terminal 1,2,3.



Consider terminal 1 and 2, Let us find equivalent impedance between 1 and 2.



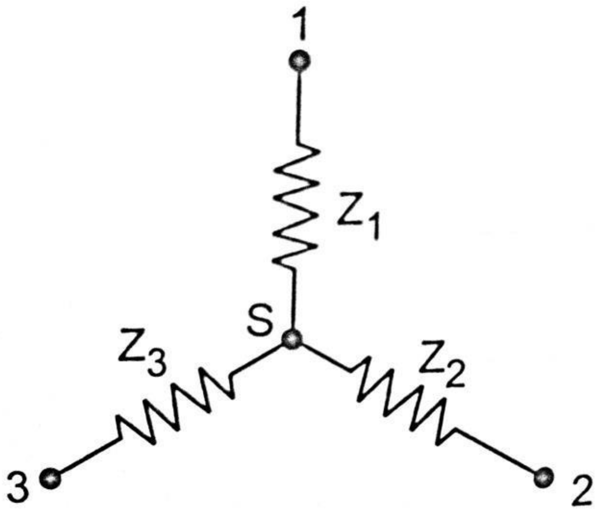
We can redraw the circuit as viewed from 1 & 2, without considering 3.

Between terminals 1 & 2, impedance is given by,

$$Z_{12} = Z_B \,||\, (Z_A + Z_C)$$

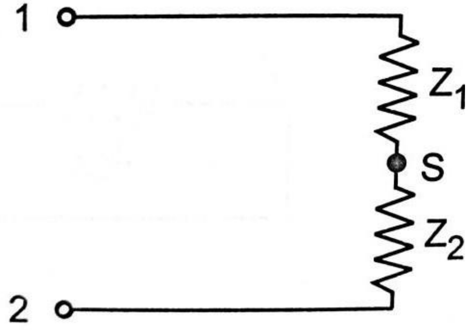
$$Z_{12} = \frac{Z_B \cdot (Z_A + Z_C)}{Z_B + (Z_A + Z_C)} \text{ --------------- } (1)$$

$$Z_{12} = \frac{Z_B \cdot (Z_A+Z_C)}{Z_B + (Z_A+Z_C)} \text{ --------------- (1)}$$



Now consider the same 2 terminals of equivalent star connection as shown.

Between 1 & 2, the impedance is $Z_{12} = Z_1 + Z_2$ ------------------------(2)



To have star connection equivalent to delta connection,

It is necessary that impedances are calculated between terminals 1&2 in both cases should be equal and hence equate (1) & (2).

$$\frac{Z_B \cdot (Z_A+Z_C)}{Z_B + Z_A + Z_C} = Z_1 + Z_2 \text{ --------------------(3)}$$

Similarly,

$$\frac{Z_C \cdot (Z_A+Z_B)}{Z_C + Z_A + Z_B} = Z_2 + Z_3 \text{ --------------------(4)}$$

$$\frac{Z_A \cdot (Z_B+Z_C)}{Z_A + Z_B + Z_C} = Z_1 + Z_3 \text{ --------------------(5)}$$

To find Z_1, Z_2 & Z_3 in terms of Z_A, Z_B & Z_C

Subtract equation (4) from (3)

$$\frac{Z_B \cdot (Z_A+Z_C)}{Z_B + (Z_A+Z_C)} - \frac{Z_C \cdot (Z_A+Z_B)}{Z_C + (Z_A+Z_B)} = Z_1 + \cancel{Z_2} - \cancel{Z_2} - Z_3$$

$$\frac{Z_A Z_B + \cancel{Z_B Z_C} - Z_A Z_C - \cancel{Z_B Z_C}}{Z_A + Z_B + Z_C} = Z_1 - Z_3$$

$$\frac{Z_A Z_B - Z_A Z_C}{Z_A + Z_B + Z_C} = Z_1 - Z_3 \quad \text{-------------------------------(6)}$$

Adding equation (5) from (6)

$$\frac{Z_A \cdot (Z_B+Z_C)}{Z_A + (Z_B+Z_C)} + \frac{Z_A Z_B - Z_A Z_C}{Z_A + Z_B + Z_C} = Z_1 + \cancel{Z_3} + Z_1 - \cancel{Z_3}$$

$$\frac{\cancel{Z_A Z_C} + Z_A Z_B + Z_A Z_B - \cancel{Z_A Z_C}}{Z_A + Z_B + Z_C} = 2Z_1$$

$$\frac{2Z_A Z_B}{Z_A + Z_B + Z_C} = 2Z_1$$

$$\frac{Z_A Z_B}{Z_A + Z_B + Z_C} = Z_1$$

$$Z_1 = \frac{Z_A Z_B}{Z_A + Z_B + Z_C}$$

Similarly,

$$Z_2 = \frac{Z_B Z_C}{Z_A + Z_B + Z_C}$$

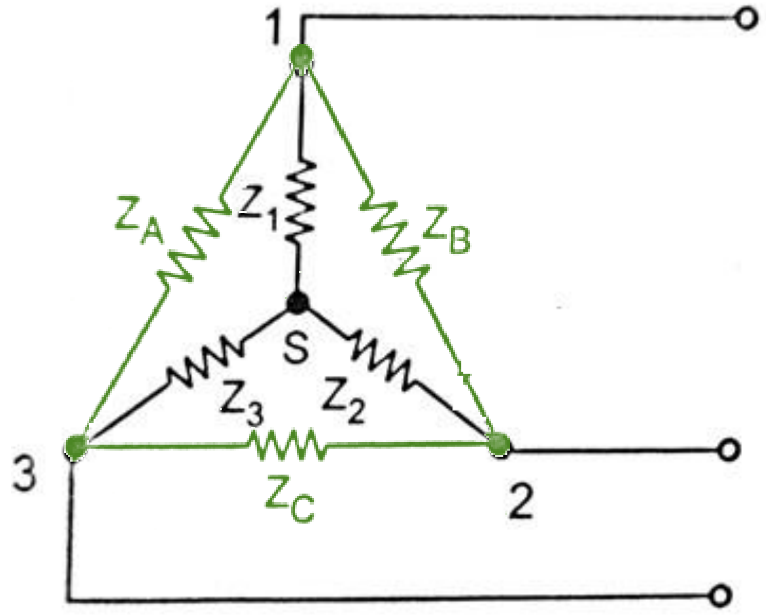
$$Z_3 = \frac{Z_C Z_A}{Z_A + Z_B + Z_C}$$

Star- Delta / T to π Transformation

Consider 3 impedances Z_1, Z_2 & Z_3 connected in Star as shown.

The terminals between which these are connected in delta are named as 1,2,3.

It is always possible to replace these Star by 3 equivalent delta connected impedances Z_A, Z_B & Z_C between the same terminal 1,2,3.



To find Z_A, Z_B & Z_C in terms of Z_1, Z_2 & Z_3.

Use the equations results from delta-star transformation

$$Z_1 = \frac{Z_A Z_B}{Z_A + Z_B + Z_C} \quad \text{----------------(1)}$$

$$Z_2 = \frac{Z_B Z_C}{Z_A + Z_B + Z_C} \quad \text{----------------(2)}$$

$$Z_3 = \frac{Z_C Z_A}{Z_A + Z_B + Z_C} \quad \text{----------------(3)}$$

Multiply (1)&(2), (2)&(3) and (3)&(1)

$$Z_1Z_2 = \frac{Z_AZ_B{}^2Z_C}{(Z_A + Z_B + Z_C)^2} \quad \text{-----------------(4)}$$

$$Z_2Z_3 = \frac{Z_BZ_C{}^2Z_A}{(Z_A + Z_B + Z_C)^2} \quad \text{-----------------(5)}$$

$$Z_3Z_1 = \frac{Z_CZ_A{}^2Z_B}{(Z_A + Z_B + Z_C)^2} \quad \text{-----------------(6)}$$

Adding (4), (5) and (6)

$$Z_1Z_2 + Z_2Z_3 + Z_3Z_1 = \frac{Z_AZ_B{}^2Z_C + Z_BZ_C{}^2Z_A + Z_CZ_A{}^2Z_B}{(Z_A + Z_B + Z_C)^2}$$

$$= \frac{Z_AZ_BZ_C(Z_A + Z_B + Z_C)}{(Z_A + Z_B + Z_C)^2}$$

$$Z_1Z_2 + Z_2\,Z_3 + Z_3\,Z_1 = \frac{Z_AZ_B\ \ Z_C}{(Z_A + Z_B + Z_C)} \quad \text{---------------------------(7)}$$

$$Z_1Z_2 + Z_2 Z_3 + Z_3 Z_1 = \frac{Z_A Z_B \ Z_C}{(Z_A + Z_B + Z_C)} \quad \text{---------------------------(7)}$$

From equation (1)

$$Z_1 = \frac{Z_A Z_B}{Z_A + Z_B + Z_C}$$

Substitute in equation (7) in R.H.S , we get

$$Z_1Z_2 + Z_2Z_3 + Z_3Z_1 = Z_1Z_C$$

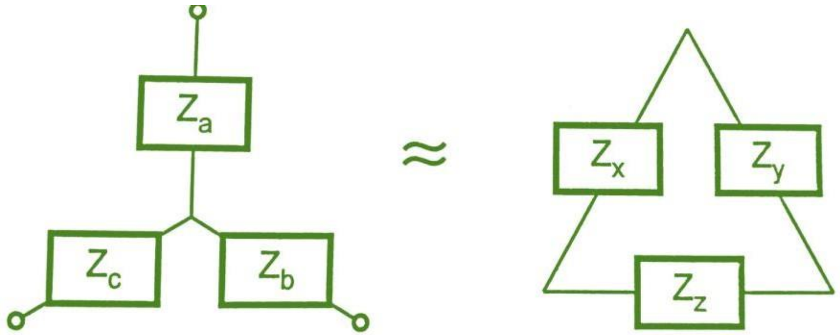
$$Z_C = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1}$$

$$Z_C = Z_2 + Z_3 + \frac{Z_2Z_3}{Z_1}$$

$$Z_A = Z_1 + Z_3 + \frac{Z_1Z_3}{Z_2}$$

$$Z_B = Z_1 + Z_2 + \frac{Z_1Z_2}{Z_3}$$

Problem 1: Three impedances $Z_a = 6\angle 90^0\ \Omega$, $Z_b = 6\angle 60^0\ \Omega$ and $Z_c = 6\angle -90^0\ \Omega$ are connected in star. Calculate the values of Z_x, Z_y and Z_z of the equivalent Delta.



$Z_a = 6\angle 90^0\ \Omega$ = 0+6j

$Z_b = 6\angle 60^0\ \Omega$ = 3+5.196j

$Z_c = 6\angle -90^0\ \Omega$ = 0-6j

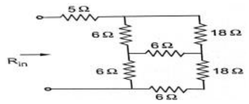
From Star to Delta Conversion,

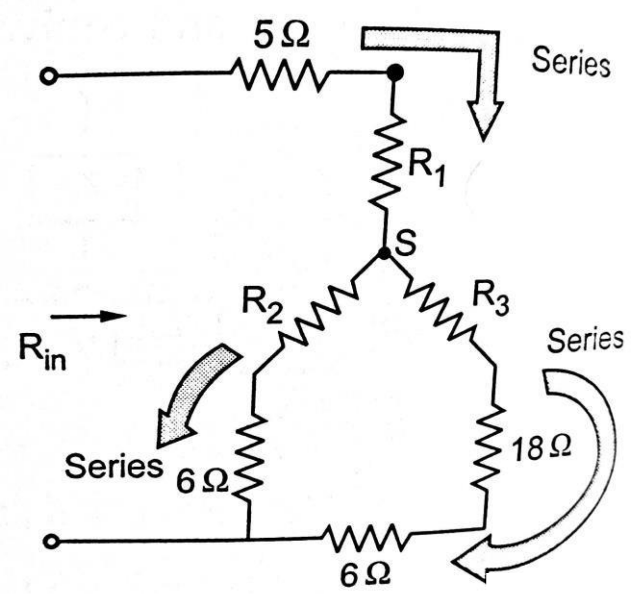
$$Z_x = Z_a + Z_c + \frac{Z_a Z_c}{Z_b} = 6j - 6j + \frac{(6j)(-6j)}{6\angle 60^0} = \frac{36}{6\angle 60^0} = 6\angle -60^0$$

$$Z_y = Z_a + Z_b + \frac{Z_a Z_b}{Z_c} = 6j + 3 + 5.196j + \frac{6\angle 90^0\ X\ 6\angle 60^0}{6\angle -90^0} = 6j = 6\angle 90^0$$

$$Z_z = Z_b + Z_c + \frac{Z_b Z_c}{Z_a} = 3 + 5.196j - 6j + \frac{6\angle 60^0\ X\ 6\angle -90^0}{6\angle 90^0} = -6j = 6\angle -90^0$$

Problem 2: **Determine R_{in} using Star- Delta transformation in the network shown.**



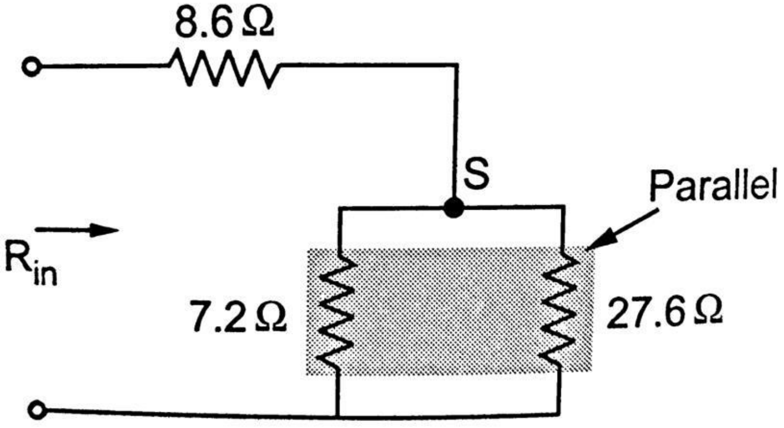


Converting upper delta of 6Ω, 6Ω and 18Ω to star,

$$R_1 = \frac{6X18}{6+6+18} = 3.6\ \Omega$$

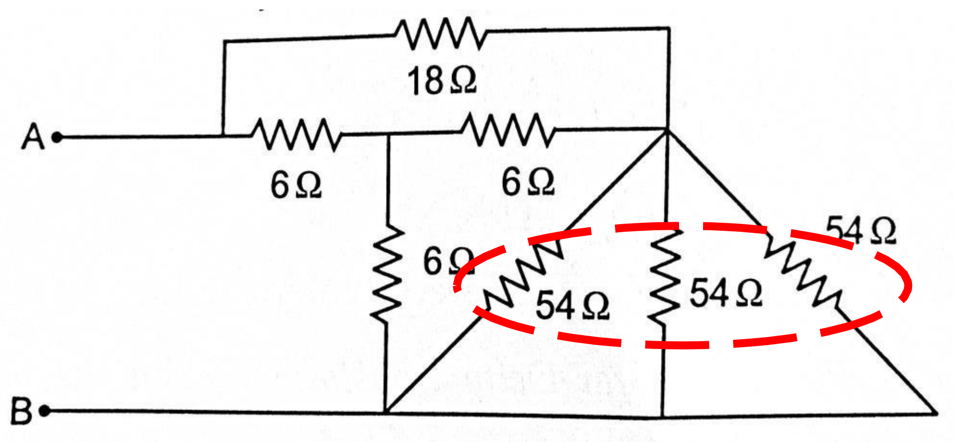
$$R_2 = \frac{6X6}{6+6+18} = 1.2\ \Omega$$

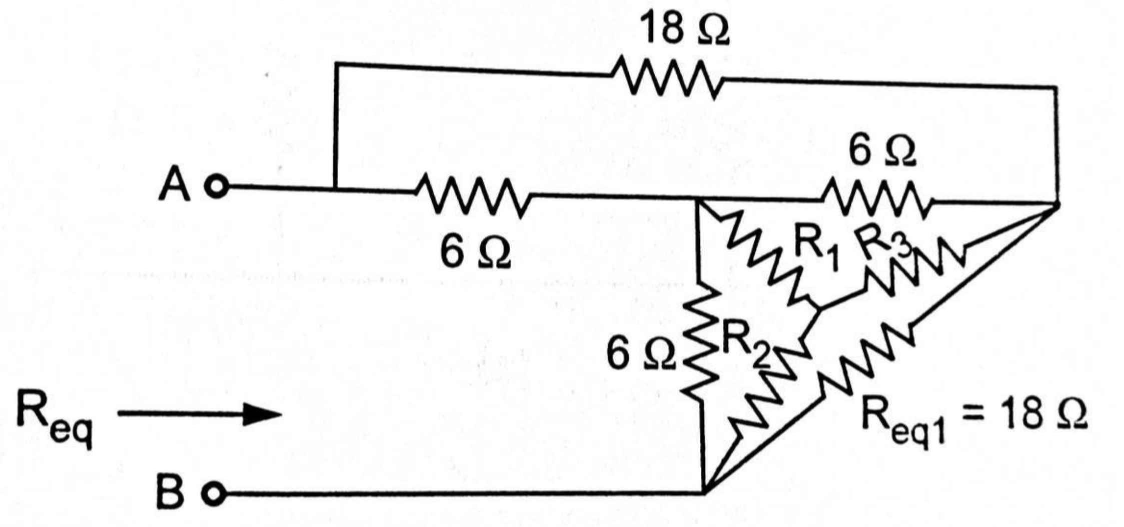
$$R_3 = \frac{6X18}{6+6+18} = 3.6\ \Omega$$



$$R_{in} = 8.6 + (7.2 \parallel 27.6)$$
$$= 8.6 + 5.7103$$
$$= 14.3103\ \Omega$$

Problem 3: **Compare the resistance across the terminals A and B of the network shown using Star-Delta transformation.**



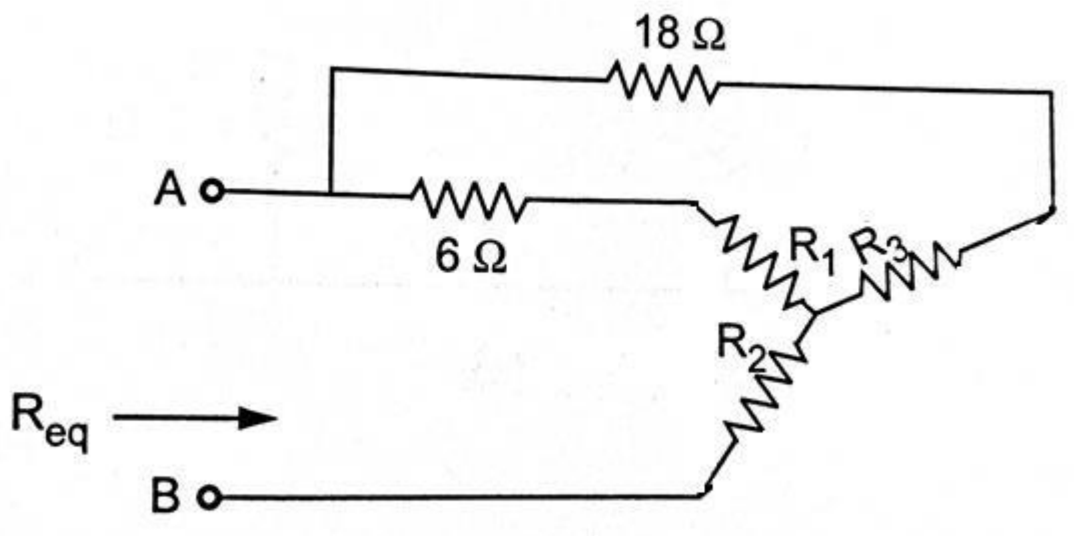


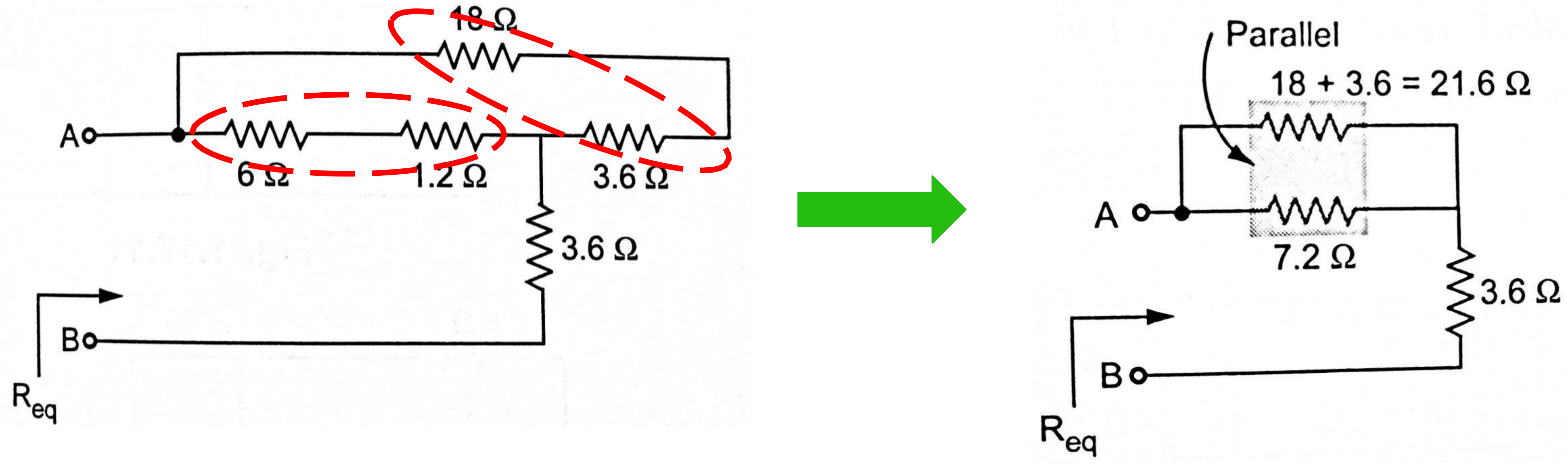
R_{eq} = 54 || 54 || 54 = 18Ω

$$R_1 = \frac{6X6}{6+6+18} = 1.2\ \Omega$$

$$R_2 = \frac{6X18}{6+6+18} = 3.6\ \Omega$$

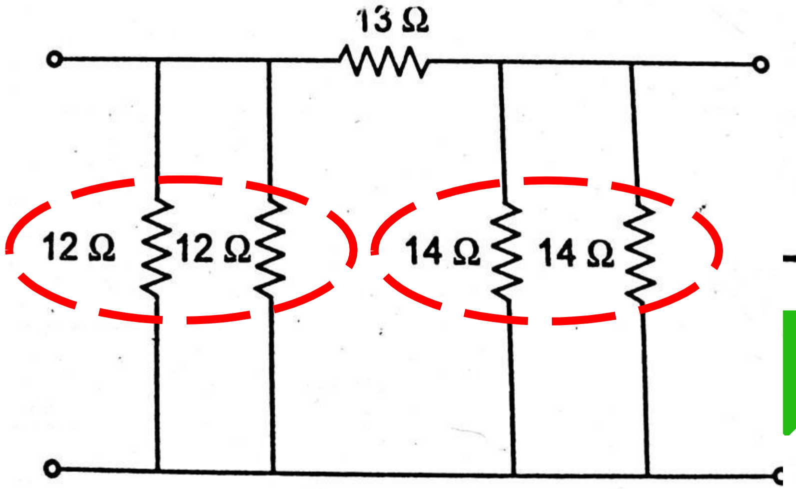
$$R_3 = \frac{6X18}{6+6+18} = 3.6\ \Omega$$

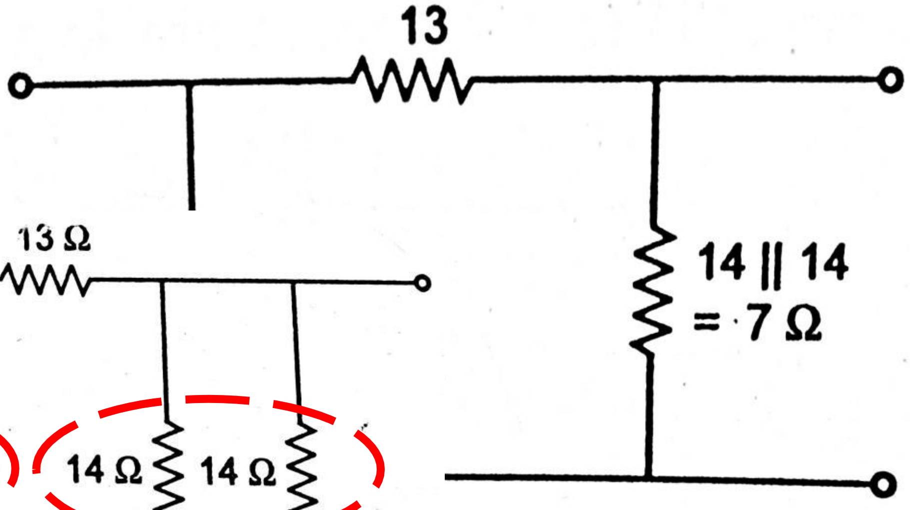


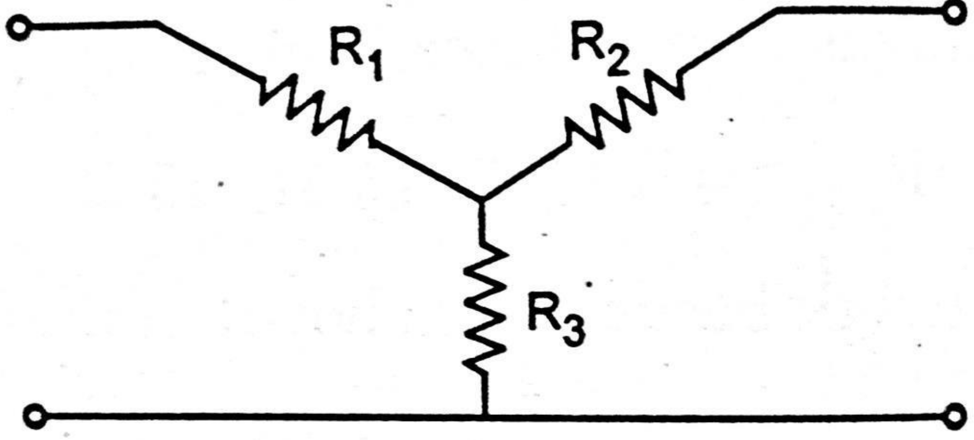


R_{eq} = 3.6 + (21.6 || 7.2) = 5.4 + 3.6 = 9 Ω

Problem 4: Find the star equivalent of the following circuit shown.





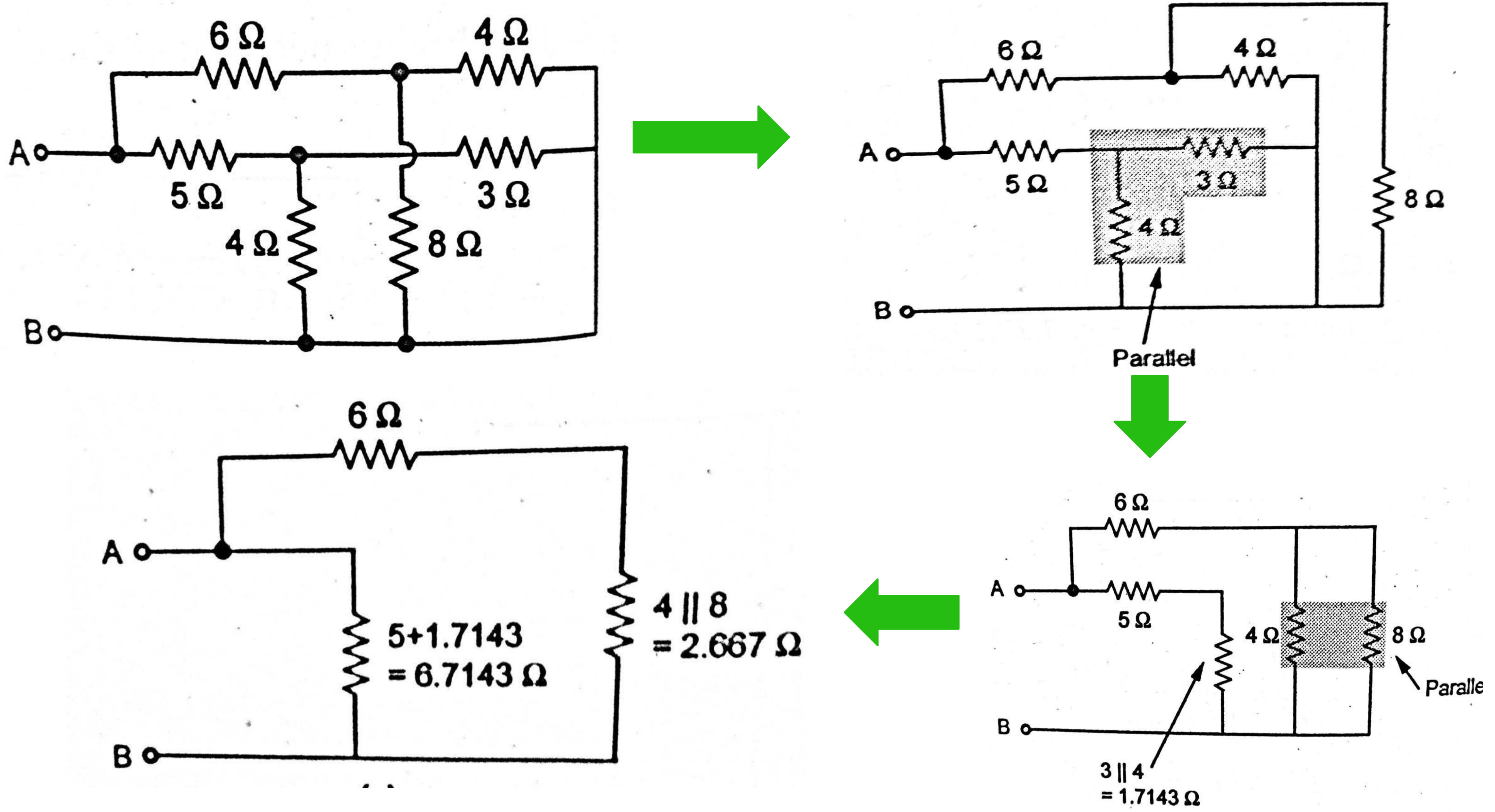


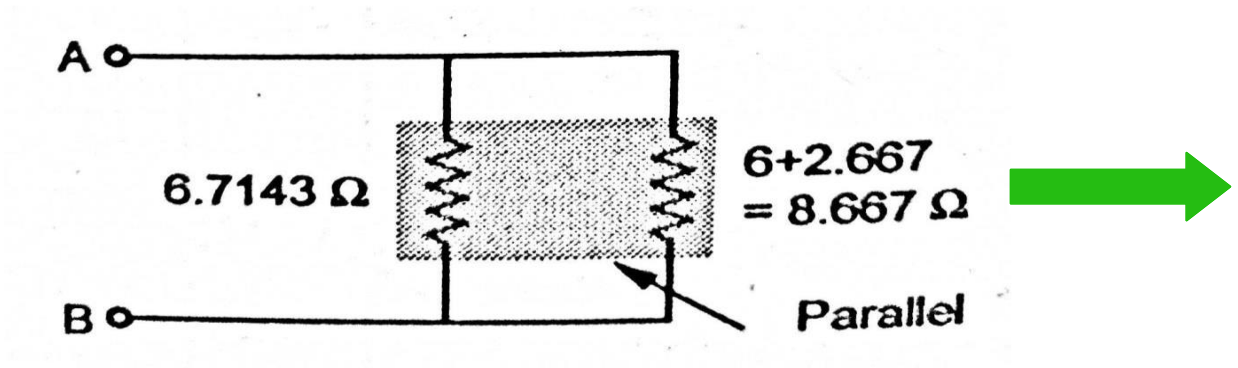
$$R_1 = \frac{13X6}{13+6+7} = 3\ \Omega$$

$$R_2 = \frac{13X7}{13+6+7} = 3.5\ \Omega$$

$$R_3 = \frac{7X6}{13+6+7} = 1.6154\ \Omega$$

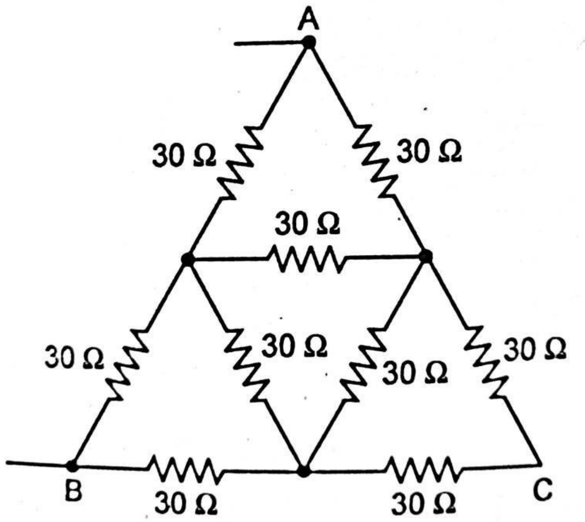
Problem 5: Using Star-delta transformation, find an equivalent resistance between A and B for the network shown.





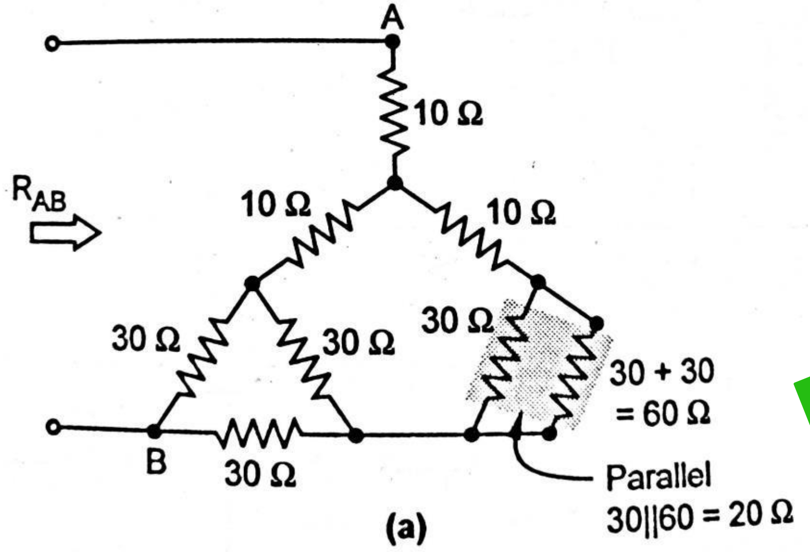
$$R_{AB} = 6.7143 \,||\, 8.667 = \frac{6.7143 \; X \; 8.667}{6.7143+8.667} = 3.7833\ \Omega$$

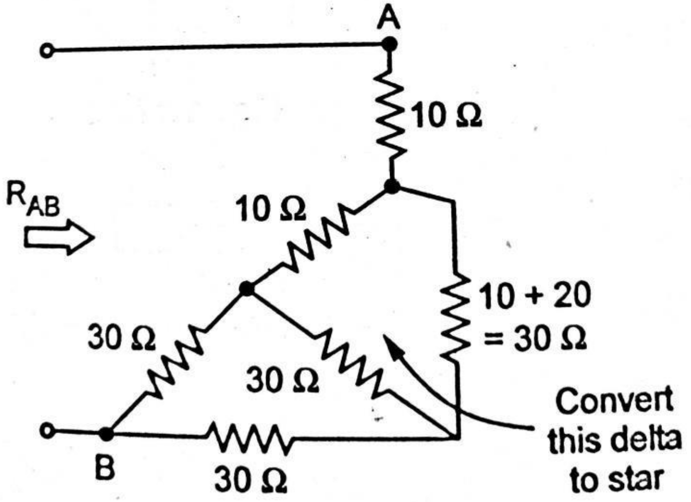
Problem 6: Find equivalent resistance at AB terminals in Figure.

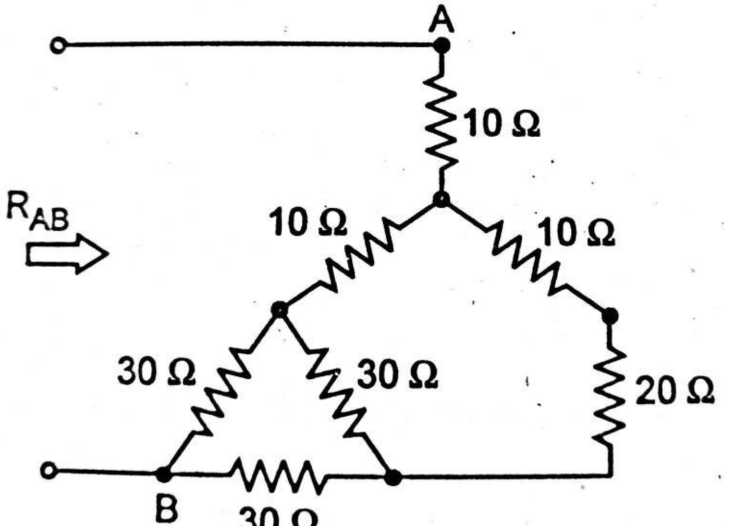


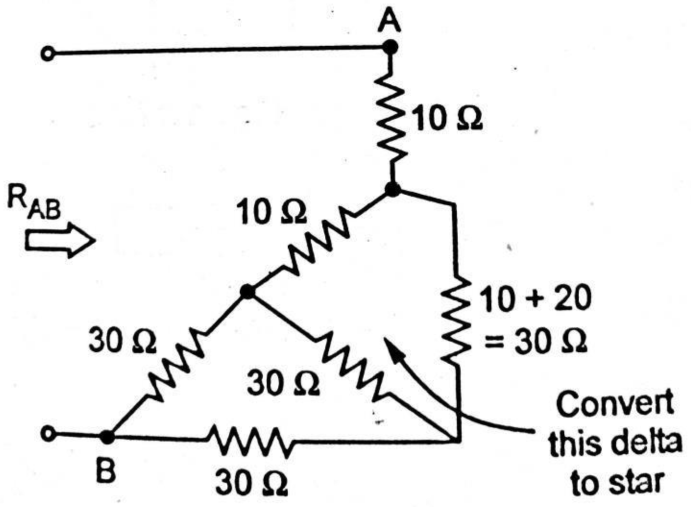
Convert 30 Ω upper delta to star.

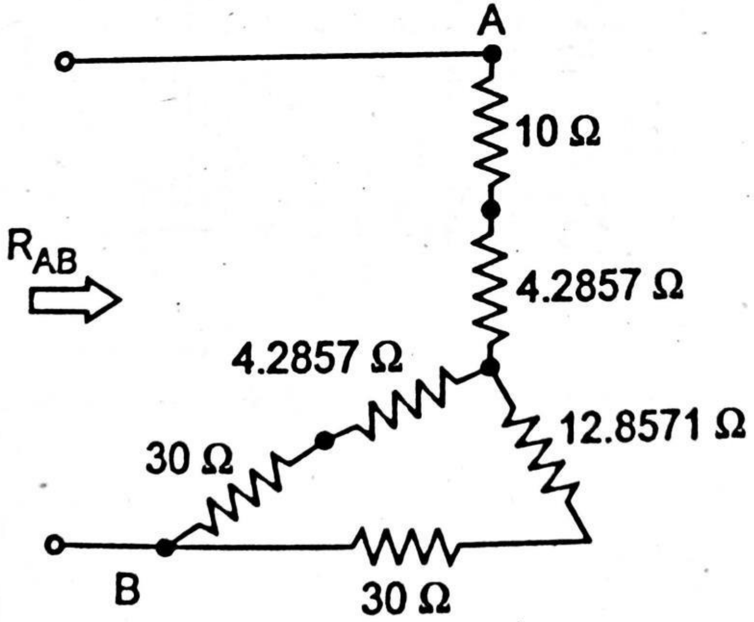
Each resistance of equivalent star is $\frac{30X30}{30+30+30}$ = 10Ω

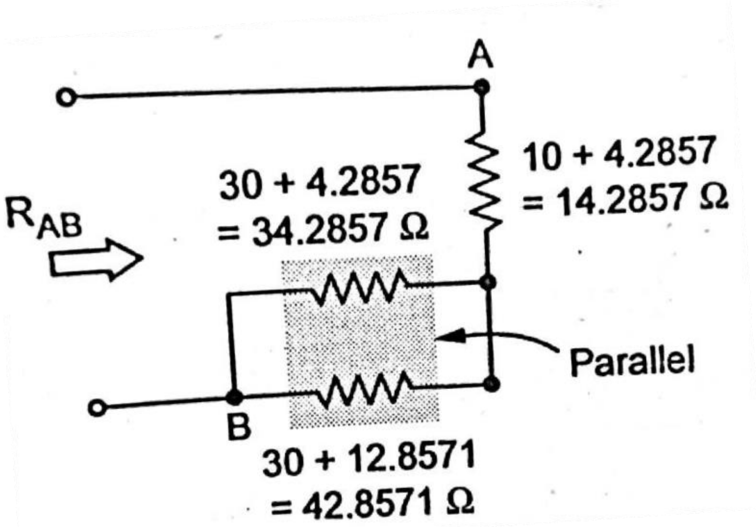


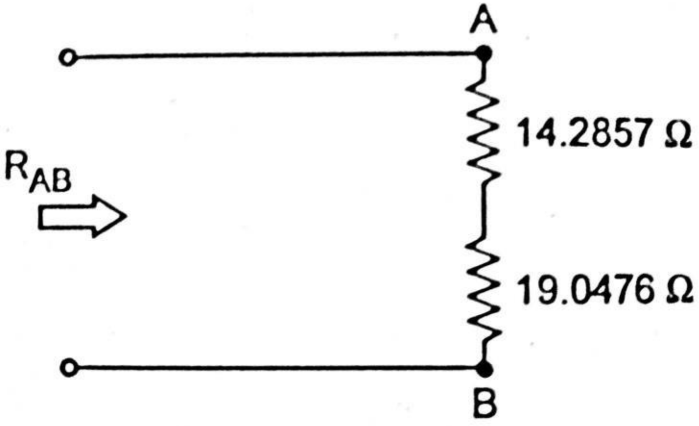






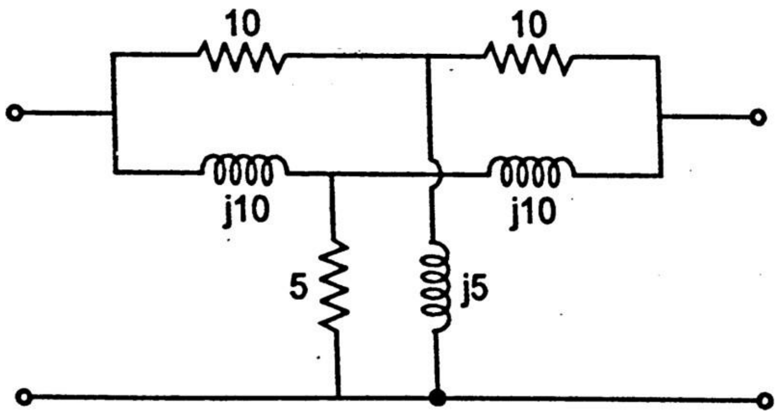


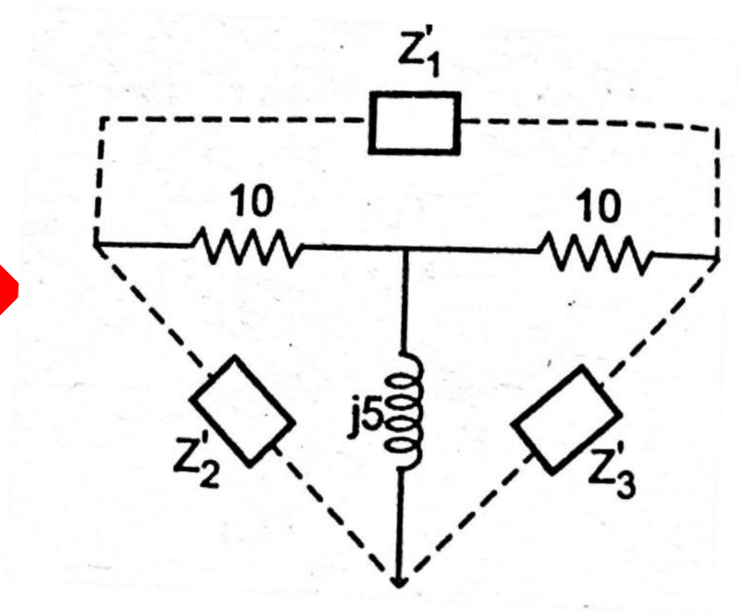




R_{AB} = 14.2857+19.0476 = 33.3333 Ω

Problem 7: The network shown consists of two star connected circuits in parallel. Obtain the single delta connected equivalent.

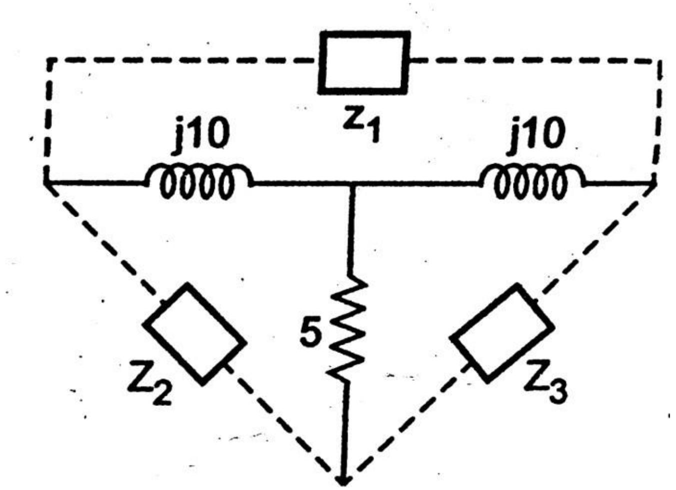




Z_1^1 = 10 + 10 + $\frac{10\ X\ 10}{5j}$ = 20-20j

Z_2^1 = 10 + 5j + $\frac{10\ X\ 5j}{10}$ = 10+10j

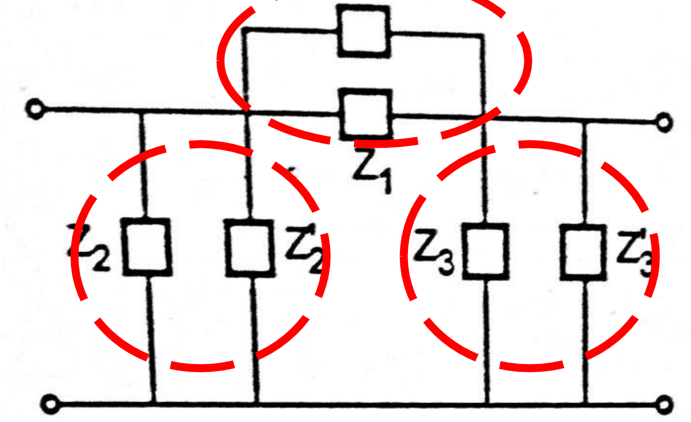
Z_3^1 = 10 + 5j + $\frac{10\ X\ 5j}{10}$ = 10+10j



Z_1 = 10j + 10j + $\frac{10j\ X\ 10j}{5}$ = -20+20j

Z_2 = 10j + 5 + $\frac{10j\ X\ 5}{10j}$ = 10+10j

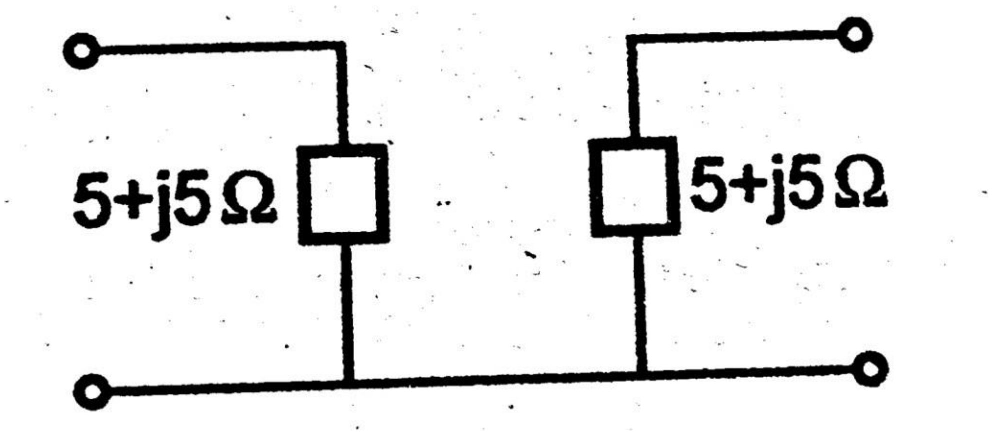
Z_3 = 10j + 5 + $\frac{10j\ X\ 5}{10j}$ = 10+10j



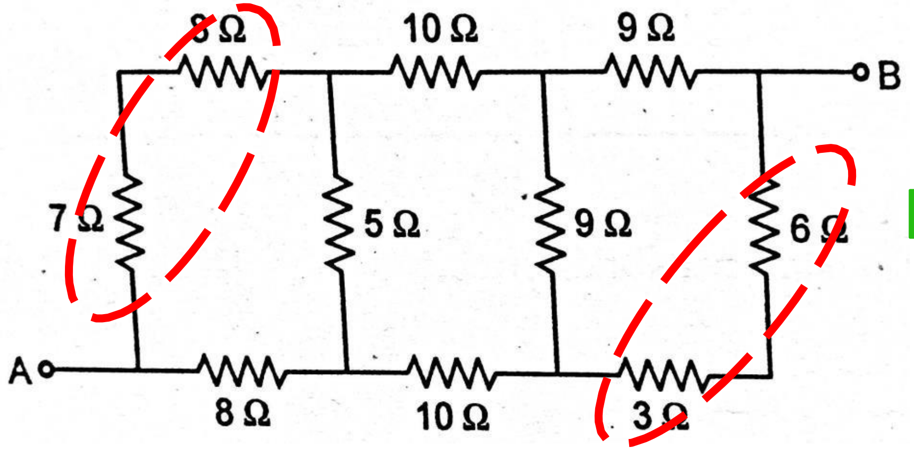
$$Z_1 \,||\, Z_1{}^1 = \frac{(-20+20j)(20-20)j}{-20+20j+20-20j} = \infty \text{ open circuit}$$

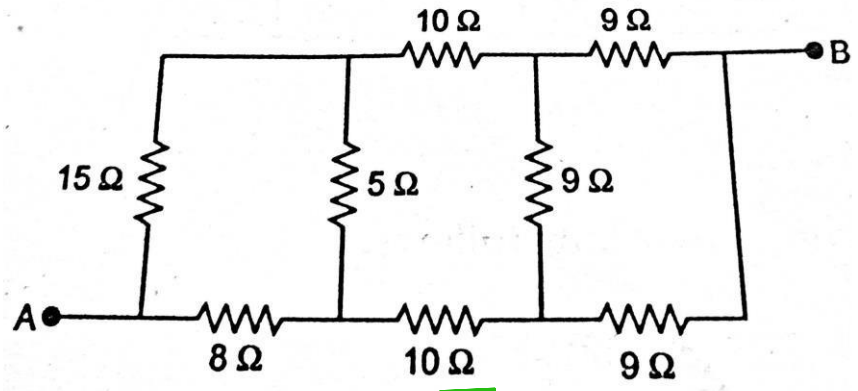
$$Z_2 \,||\, Z_2{}^1 = \frac{(10+10j)(10+10j)}{10+10j+10+10j} = 5+5j$$

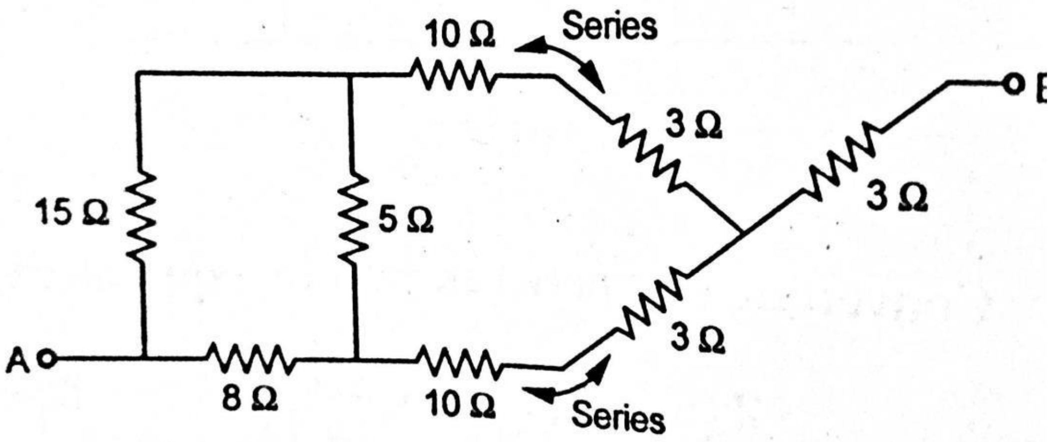
$$Z_3 \,||\, Z_3{}^1 = \frac{(10+10j)(10+10j)}{10+10j+10+10j} = 5+5j$$

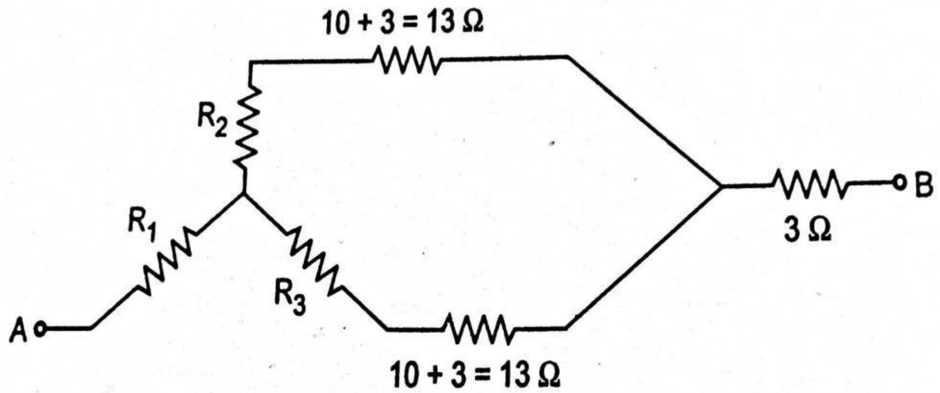


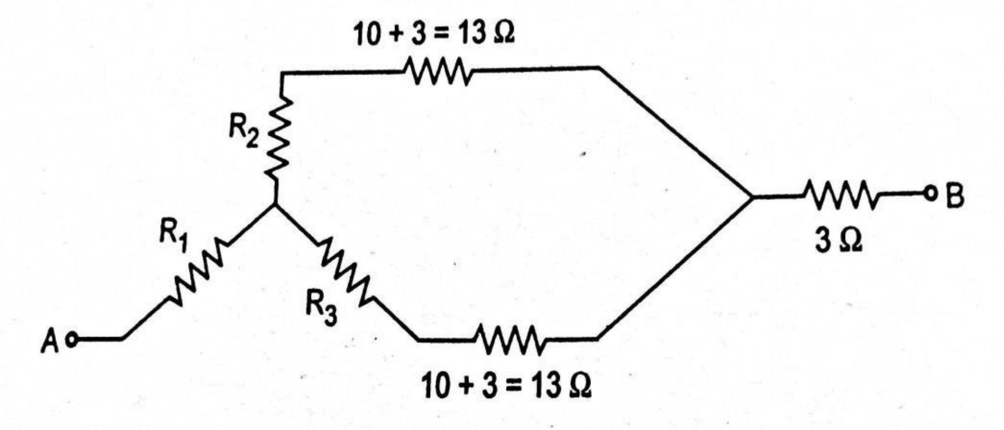
Problem 8: **Find the equivalent resistance between the terminal A and B in the network shown in figure. Using star-delta transformation.**







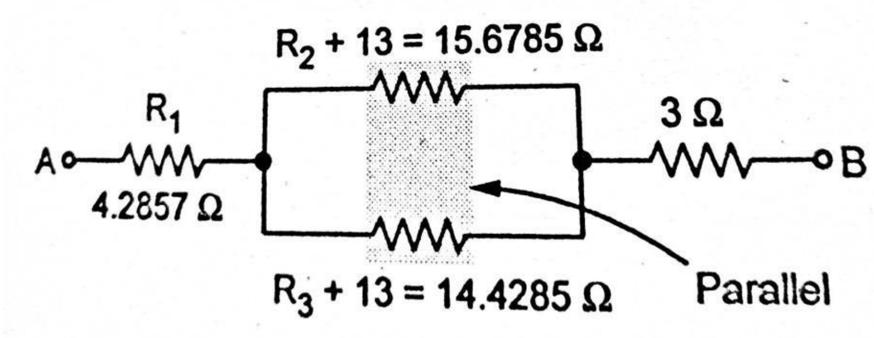


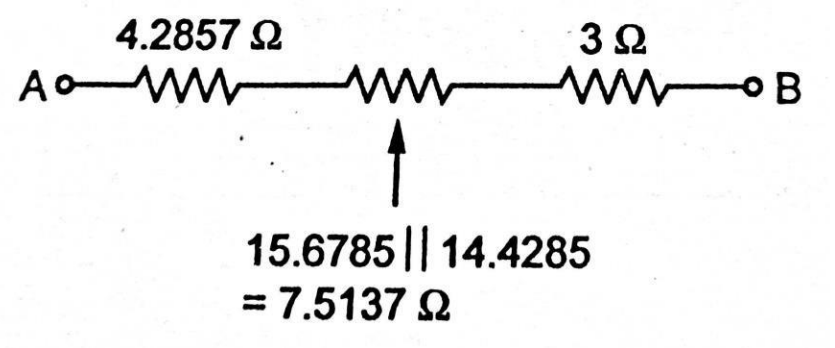


$$R_1 = \frac{15X8}{15+8+5} = 4.2857\ \Omega$$

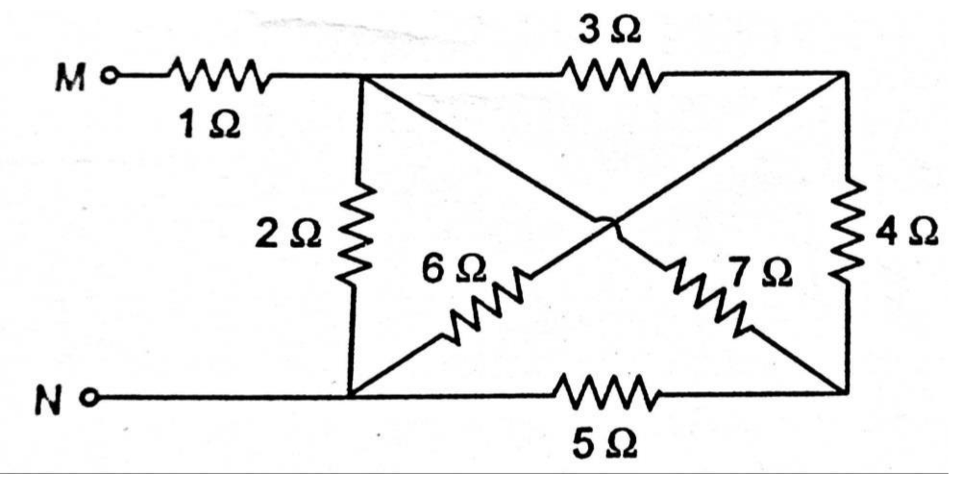
$$R_2 = \frac{15X5}{15+8+5} = 2.6785\ \Omega$$

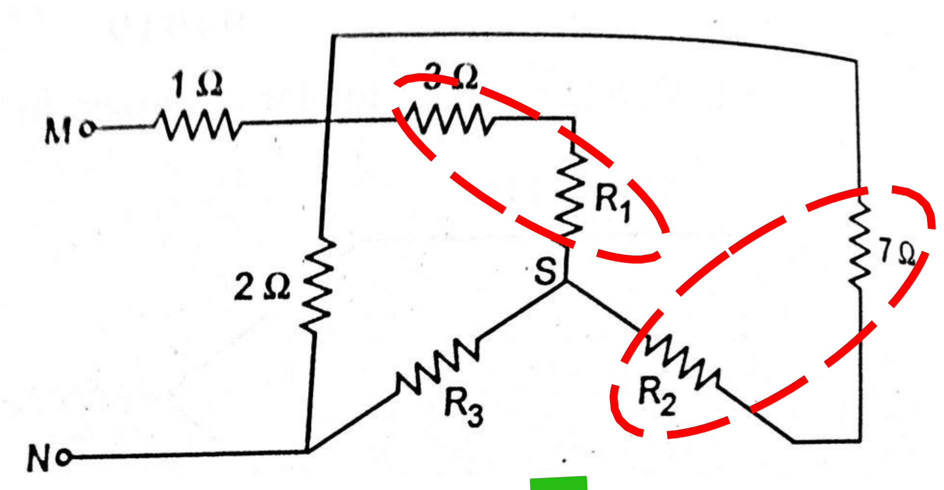
$$R_3 = \frac{5X8}{15+8+5} = 1.4285\ \Omega$$





Problem 9: Using star-delta transformation, determine the resistance between M and N of the network shown.

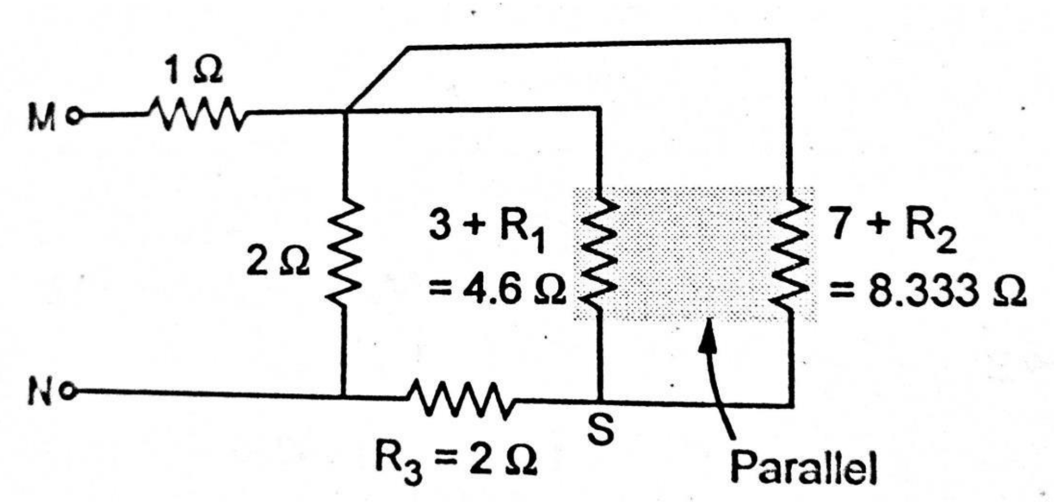


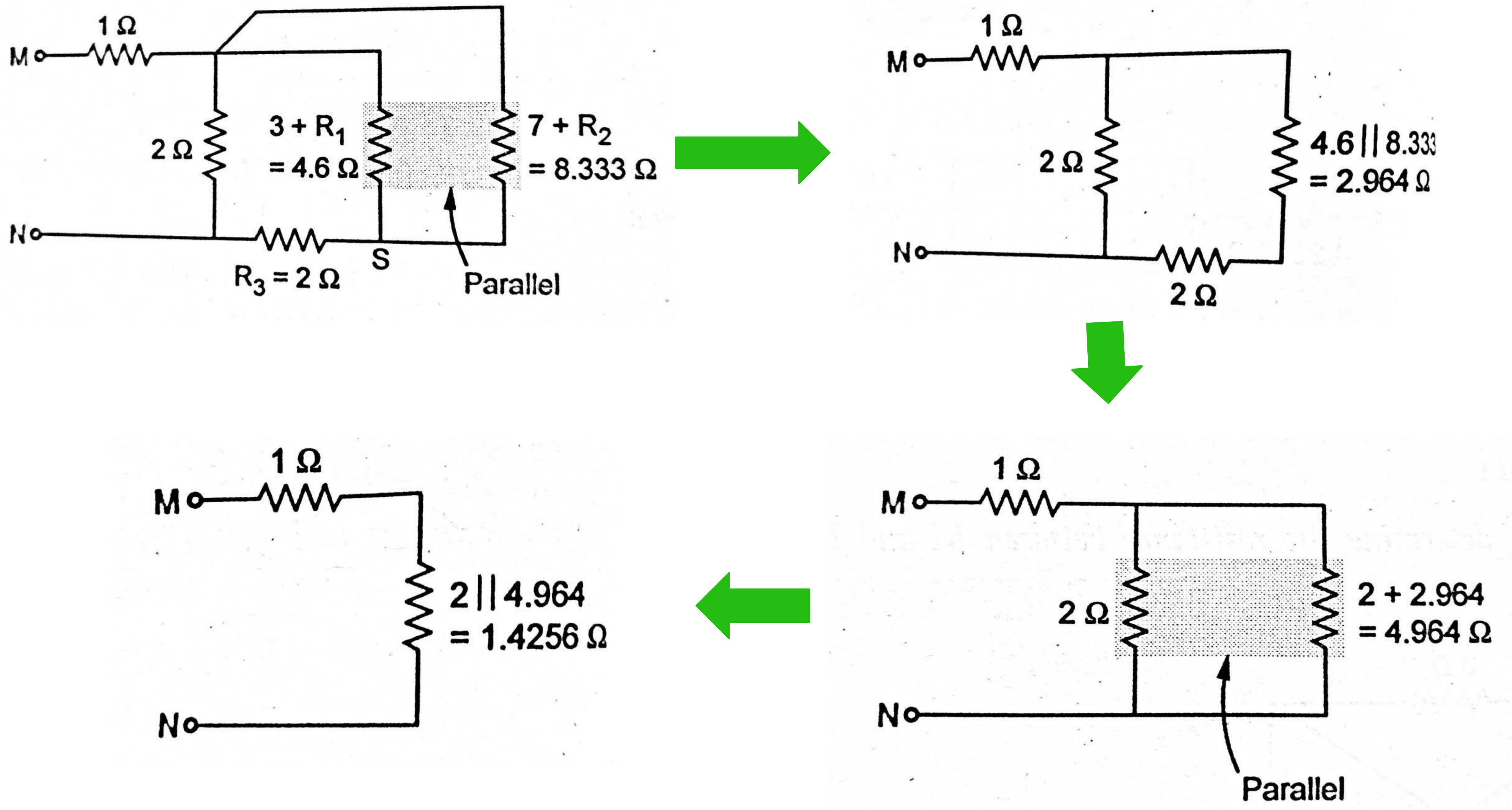


$$R_1 = \frac{6X4}{6+4+5} = 1.6\ \Omega$$

$$R_2 = \frac{5X4}{6+4+5} = 1.333\ \Omega$$

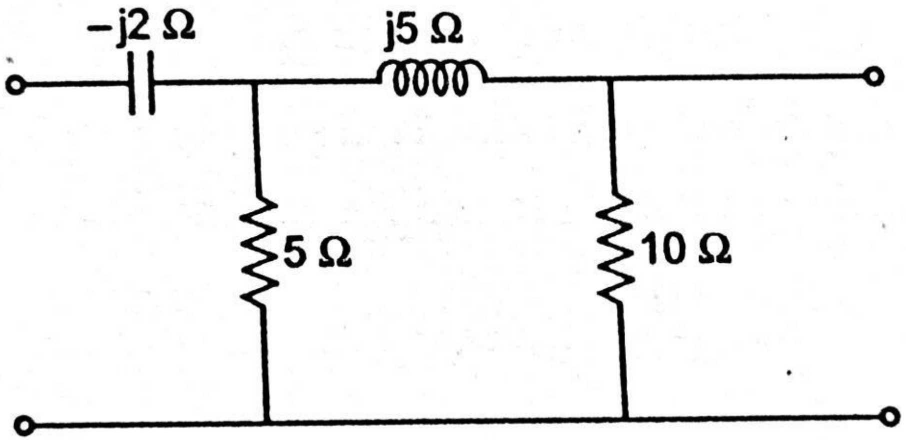
$$R_3 = \frac{6X5}{6+4+5} = 2\ \Omega$$

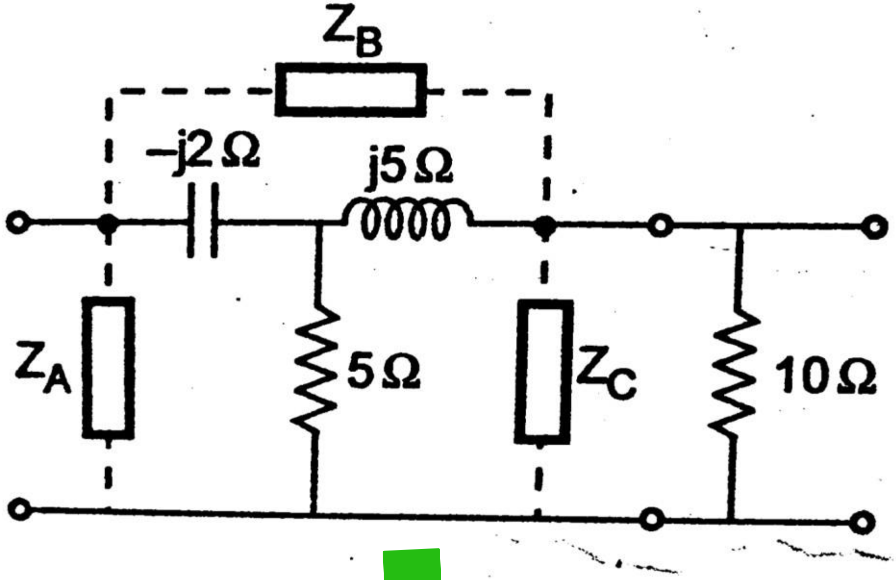




$$R_{MN} = 1 + 1.4256 = 2.4256\ \Omega$$

Problem 10: Obtain delta connected equivalent of the network shown.

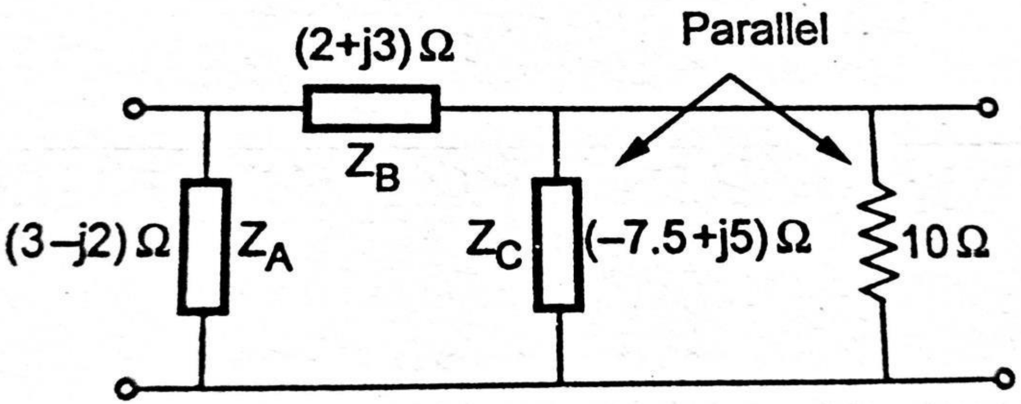


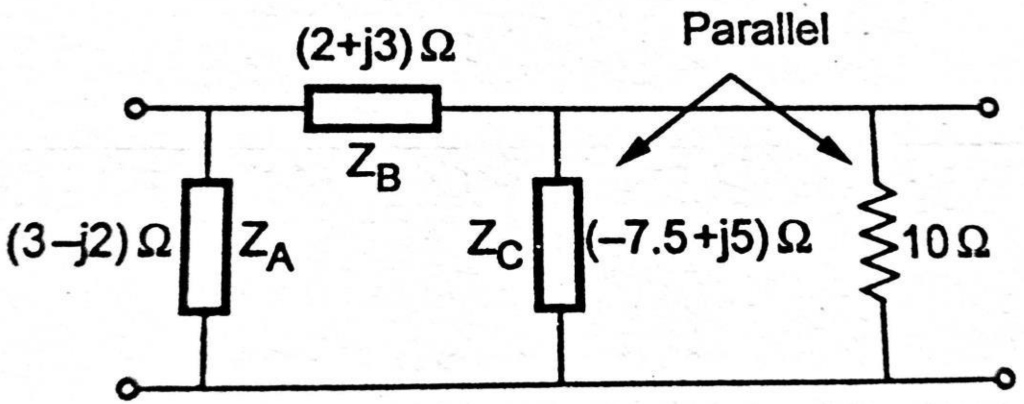


$$Z_A = -2j + 5 + \frac{(-2j)\ X\ 5}{5j} = 3\text{-}2j\ \Omega$$

$$Z_B = -2j + 5j + \frac{(-2j)\ X\ 5j}{5} = 2+3j\ \Omega$$

$$Z_C = 5j + 5 + \frac{5j\ X\ 5}{-2j} = -7.5+5j\ \Omega$$

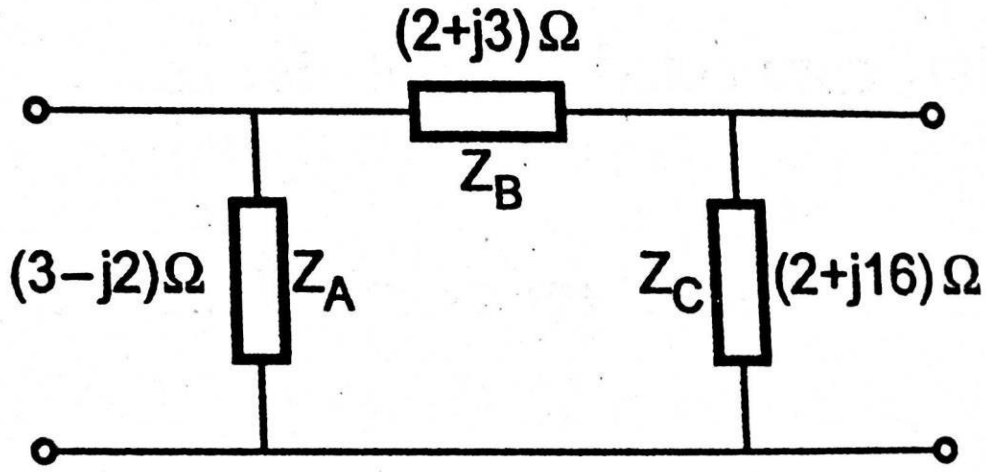




It is clear that Z_C and 10 Ω are in parallel. Hence

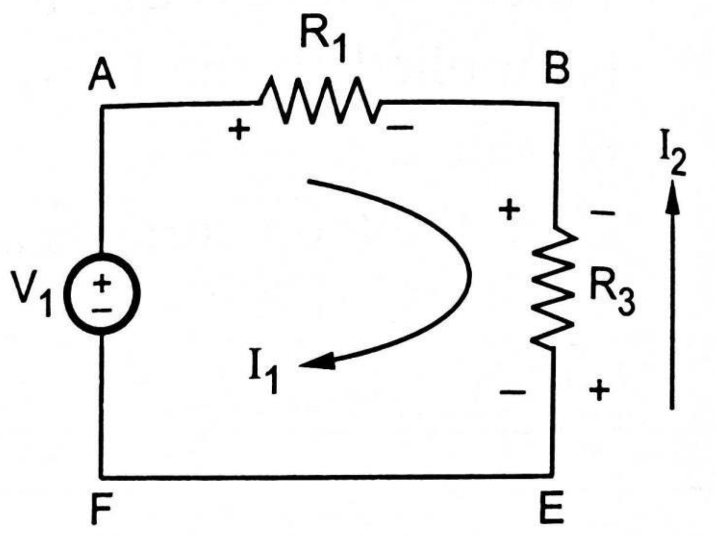
$$Z'_C = Z_C \;||\; 10 = \frac{(-7.5+5j)(10)}{-7.5+5j+10} = \frac{(-7.5+5j)(10)}{2.5+5j}$$

= 16.1245 ∠ 82.87 = (2 + 16j) Ω



Loop Analysis

Consider loop A-B-E-F-A,



For branch B-E, polarities of voltage drops will be B +ve, E –ve for current I_1 while E +ve, B –ve for current I_2 flowing through R3.

While writing loop equations, assume main loop current as positive and remaining loop current must be treated as negative for common branches.

Loop equations for the network are

For loop A-B-E-F-A,

$$-I_1R_1 - I_1R_3 + I_2R_3 + V_1 = 0$$

Points to remember for Loop analysis

1. While assuming loop currents make sure that at least one loop current links with every element.

2. No two loops must be identical.

3. Choose minimum number of loop currents.

4. Convert current sources if present, into their equivalent voltage sources for loop analysis, whenever possible.

5. If current in a particular branch is required, then try to choose loop current in such a way that only one loop current links with branch.

Steps for Loop analysis

1. Choose the various loops.

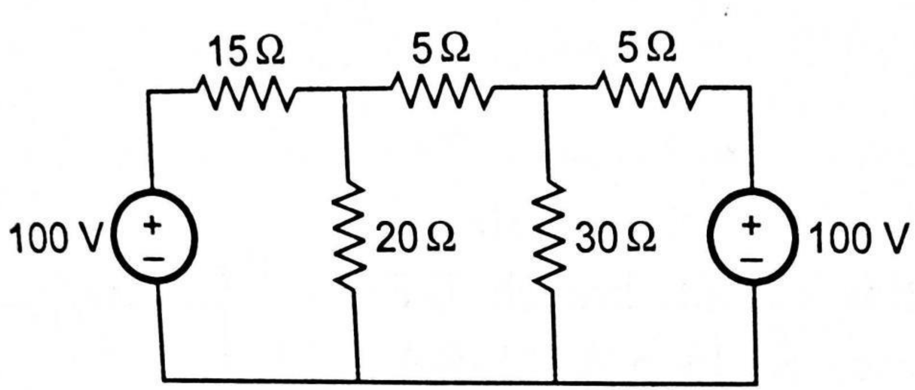
2. Show the various loop currents and the polarities of associated voltage drops.

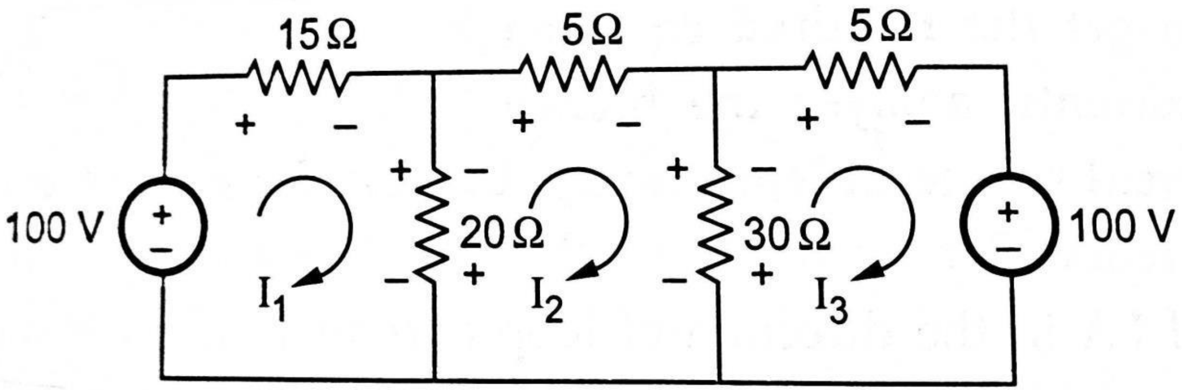
3. Before applying KVL, look for any current source. Analyse the branch consisting current source independently and express the current source value interms of assumed loop currents. Repeat this for all the current sources.

4. After the step 3, apply KVL to those loops, which do not include any current source. A loop cannot be defined through current source from KVL point of view. Follow the sign convention.

5. Solve the equations obtained in step 3 and step 4 simultaneously, to obtain required unknowns.

Problem 1: **For the circuit shown, find the current through 30Ω resistance using mesh analysis.**





Apply KVL to the various loops,

Loop 1, $$-15I_1 - 20I_1 + 20I_2 + 100 = 0$$

$$\boxed{+35I_1 - 20I_2 = 100}$$

Loop 2, $$-5I_2 - 30I_2 + 30I_3 - 20I_2 + 20I_1 = 0$$

$$\boxed{20I_1 - 55I_2 + 30I_3 = 0}$$

Loop 3, $$-5I_3 - 100 - 30I_3 + 30I_2 = 0$$

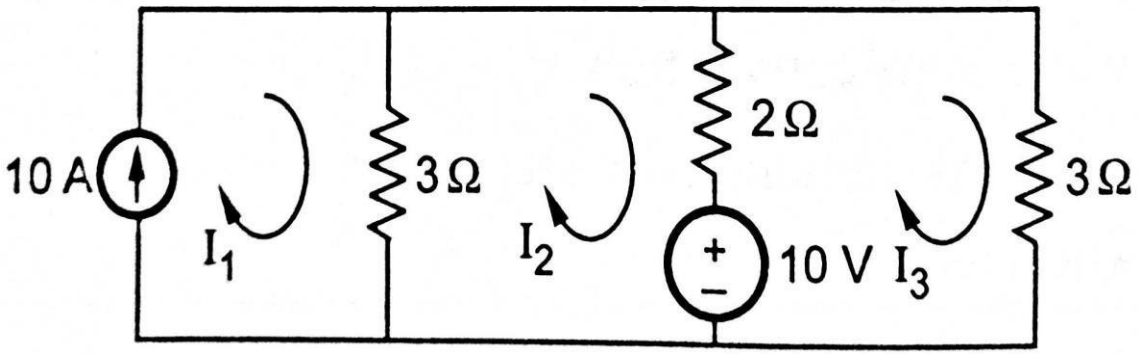
$\therefore$ $$\boxed{30I_2 - 35I_3 = 100}$$

Solving, $I_2 = -1.6$ A and $I_3 = -4.2285$ A

$\therefore$ $$I_{30\,\Omega} = I_2 - I_3 = -1.6 - (-4.2285) = \mathbf{2.6285\ A} \downarrow$$

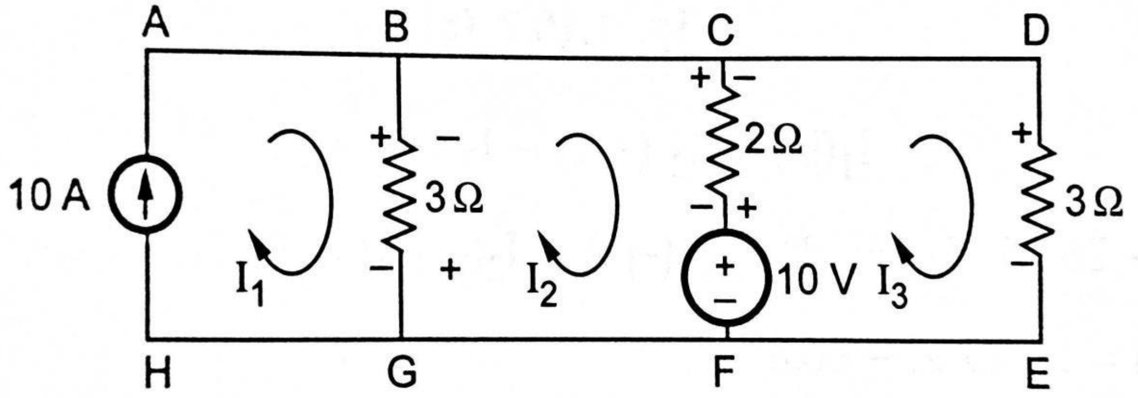
As $I_2 - I_3$ is positive, current flows in the assumed direction of I_2.

Problem 2: **Write the mesh equation for the circuit shown in figure. And determine mesh currents using mesh analysis.**



Due to the current source, before applying KVL, analyse the branch consisting of current source and express current source in terms of assumed loop currents.

I_1= 10 A ----------- (1)



Apply KVL to the loops without current source,

Loop B-C-F-G-B, -2I_2 + 2I_3 -10 -3I_2+ 3I_1 = 0

From equation (1), -5I_2+ 2I_3 = -20 -----------(2)

Loop C-D-E-F-C, -3I_3 + 10 -2I_3 + 2I_2 = 0

2I_2 - 5I_3 = -10 ------------(3)

$-5I_2 + 2I_3 = -20$ ------------(2)

$2I_2 - 5I_3 = -10$ -------------(3)

$$D = \begin{vmatrix} -5 & 2 \\ 2 & -5 \end{vmatrix} = |25 - 4| = 21$$

$$D2 = \begin{vmatrix} -20 & 2 \\ -10 & -5 \end{vmatrix} = |100 - (-20)| = 120$$

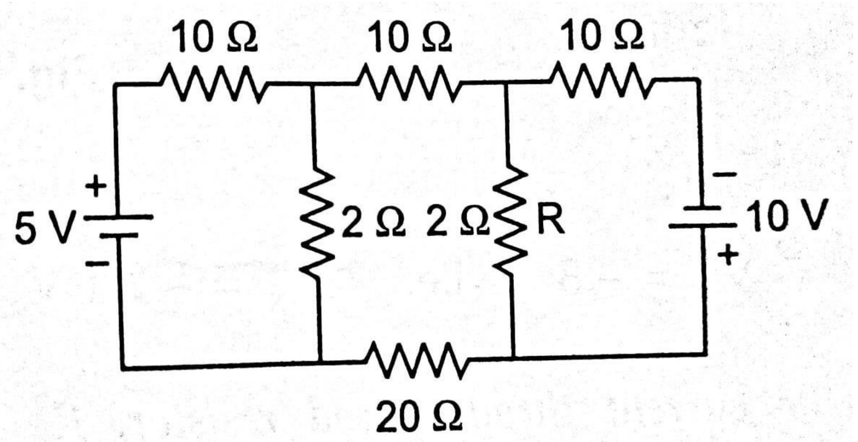
$$D3 = \begin{vmatrix} -5 & -20 \\ 2 & -10 \end{vmatrix} = |-40 - 50| = 90$$

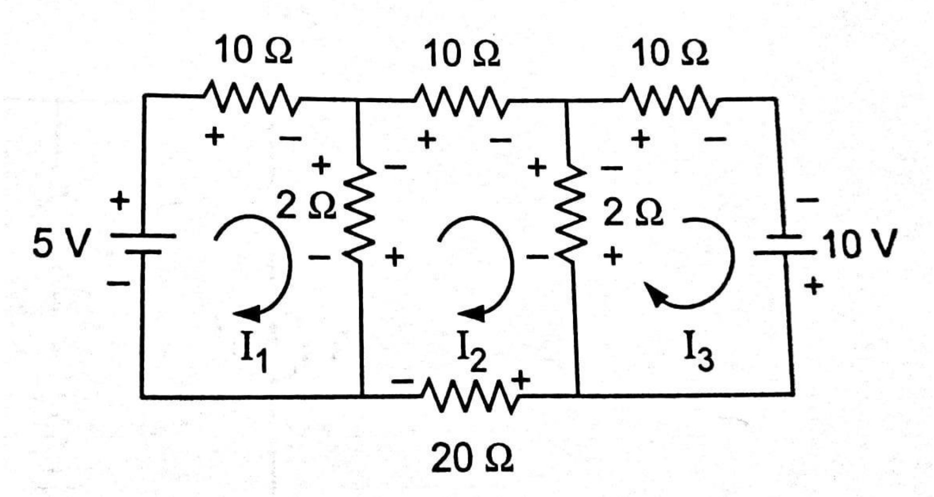
$$I_2 = \frac{D_2}{D} = \frac{120}{21} = 5.7142 \text{ A}$$

$$I_3 = \frac{D_3}{D} = \frac{90}{21} = 4.2857 \text{ A}$$

I_1 = 10 A, I_2 = 5.7142 A, I_3 = 4.2857 A

Problem 3: Find the voltage across resistance R in the network shown by mesh analysis.





Loop1, $-10I_1 - 2I_1 + 2I_2 + 5 = 0$

$-12I_1 + 2I_2 = -5$ -------------------------(1)

Loop 2, $-10I_2 - 2I_2 + 2I_3 - 20I_2 - 2I_2 + 2I_1 = 0$

$2I_1 - 34I_2 + 2I_3 = 0$ -----------------------(2)

Loop 3, $-10I_3 + 10 - 2I_3 + 2I_2 = 0$

$2I_2 - 12I_3 = -10$ ----------------------------(3)

Solving, I_1= 0.429A, I_2 = 0.075A, I_3 = 0.8458A

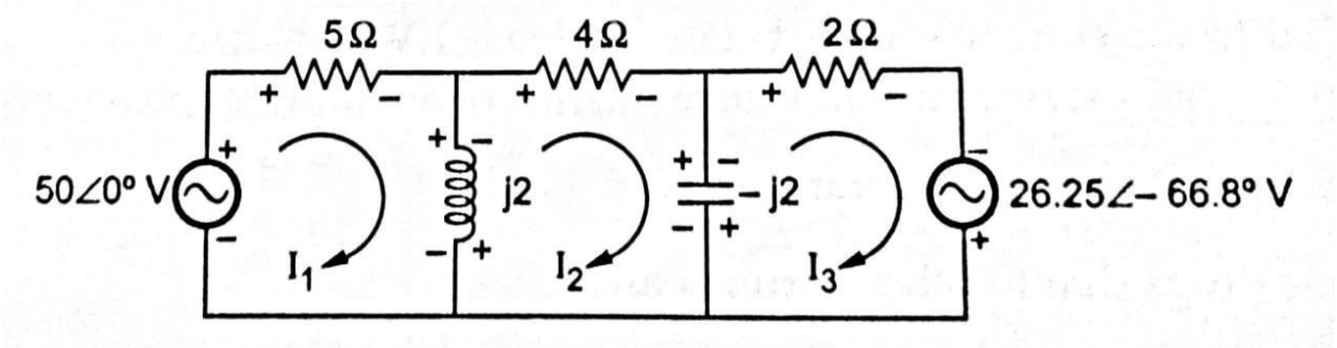
Current through R = $I_R = I_3 - I_2$ = 0.7708 A Î

Voltage across R = I_R R = 0.7708 X 2 = 1.5416 V

Problem 4: Find the current through 4Ω resistor using loop current method.

Solving, $\quad D_2 = 0$

$$I_2 = \frac{D_2}{D} = 0 \text{ A} = \text{current through } 4\ \Omega$$



$$D = \begin{vmatrix} 5+j2 & -j2 & 0 \\ j2 & -4 & -j2 \\ 0 & j2 & 2-j2 \end{vmatrix} = -84 + j24$$

and $\quad D_2 = \begin{vmatrix} 5+j2 & 50\angle 0° & 0 \\ j2 & 0 & -j2 \\ 0 & 26.25\angle -66.8° & 2-j2 \end{vmatrix}$

$$= -j2\,(2-j2)\,50\angle 0° + j2\,(5+j2)\,(26.25\angle -66.8°)$$

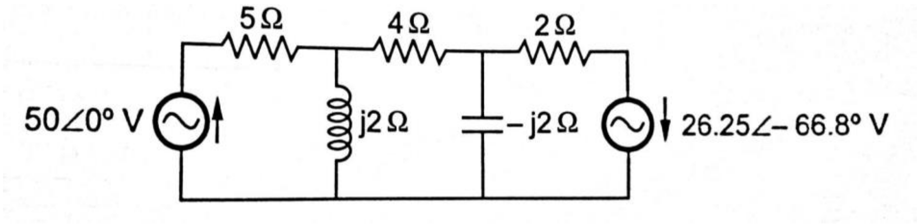
Loop1, $-5I_1 - 2jI_1 + 2jI_2 + 50\angle 0 = 0$

$(5+2j)I_1 - 2jI_2 = -50\angle 0$ ------------------ (1)

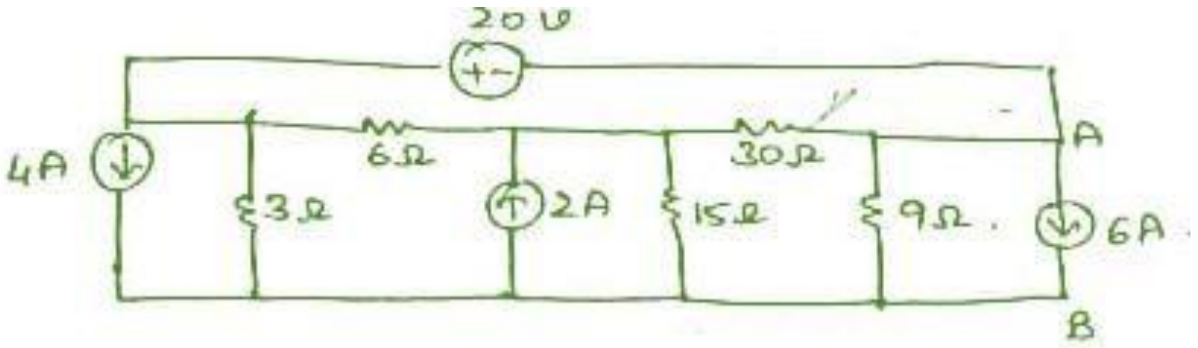
Loop 2, $-4I_2 - (-2j) + (-2j) - 2jI_2 + 2jI_1 = 0$ $2jI_1 - 4I_2 - 2jI_3 = 0$ ------------------------------- (2)

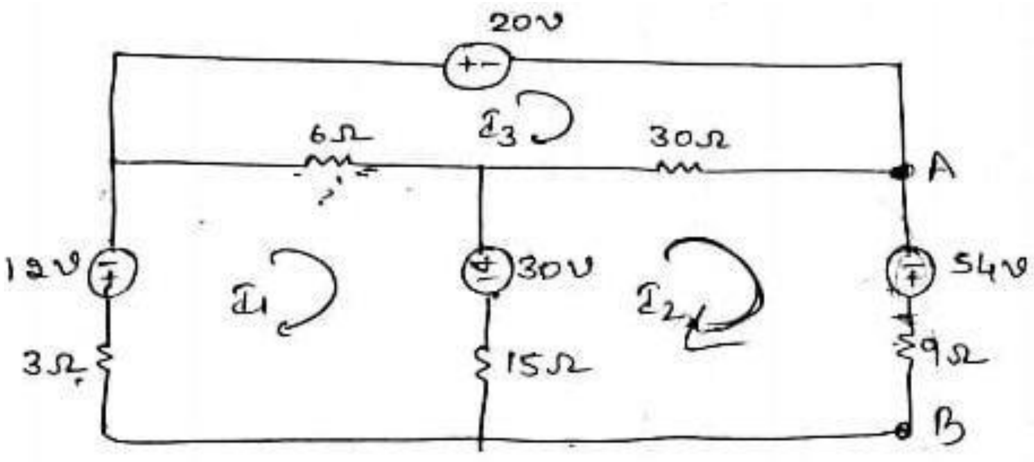
Loop 3, $-2I_3 + 26.25\angle -66.8 - (-2j) + (-2j)I_2 = 0$

$2jI_2 + (2-2j)I_3 = 26.25\angle -66.8$--------- (3)



Problem 4: **Using mesh current analysis determine the voltage across AB.**





Loop1, $-6I_1 + 6I_3 - 30 - 15I_1 + 15I_2 - 3I_1 - 12 = 0$

$-24I_1 + 15I_2 + 6I_3 = 42$ -------------------------(1)

Loop 2, $-30I_2 + 30I_3 + 54 - 9I_2 - 15I_2 + 15I_1 + 30 = 0$

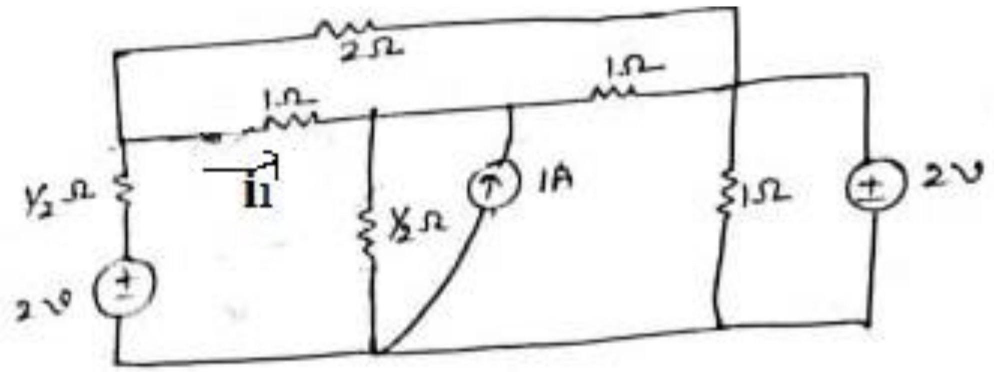
$15I_1 - 54I_2 + 30I_3 = -84$ ----------------------- (2)

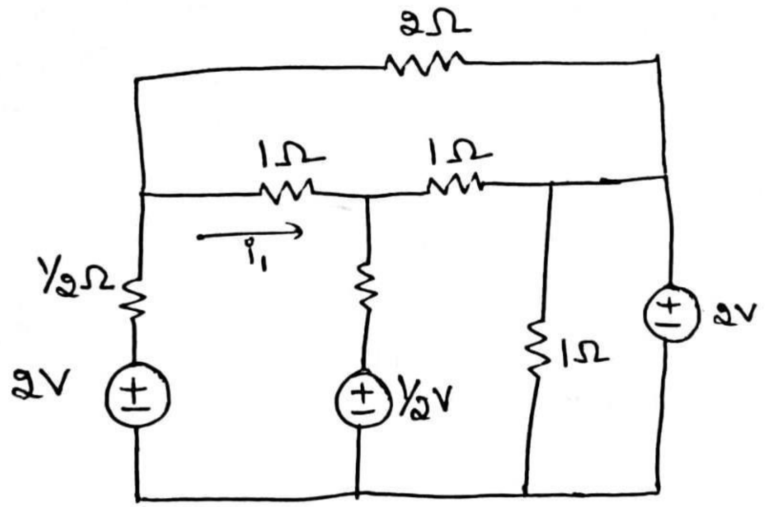
Loop 3, $-20 - 30I_3 + 30I_2 - 6I_3 + 6I_1 = 0$

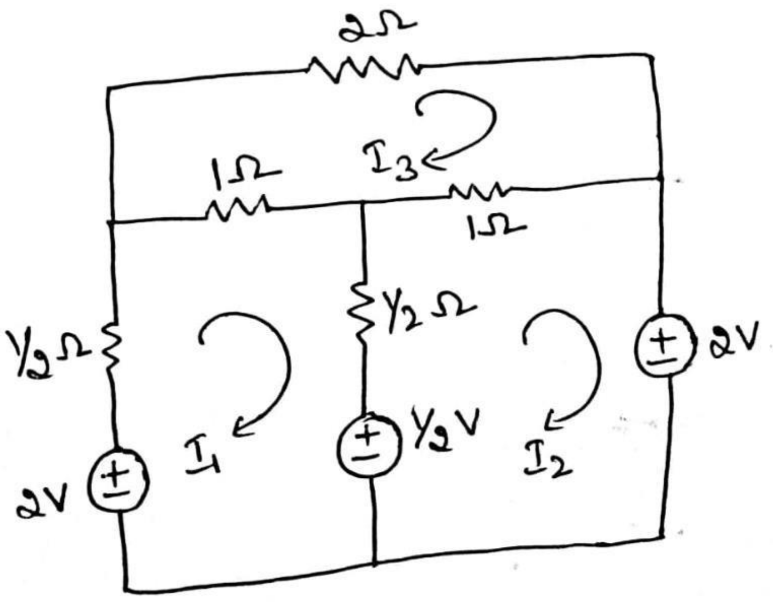
$6I_1 + 30I_2 - 36I_3 = 20$ ----------------------------(3)

Solving, I_1= 0.119A, I_2 = 2.4A, I_3 = 1.468A

Problem 5: **For the network given determine value of i.**







Loop1, $-0.5I_1 - I_1 + I_3 - 0.5I_1 + 0.5I_2 - 0.5 + 2 = 0$

$2I_1 - 0.5I_2 - I_3 = 1.5$ ------------------------ (1)

Loop 2, $-I_2 + I_3 - 2 + 0.5 - 0.5I_2 + 0.5I_1 = 0$

$0.5I_1 - 1.5I_2 + I_3 = 1.5$ ----------------------(2)

Loop 3, $-2I_3 - I_3 + I_2 - I_3 + I_1 = 0$

$I_1 + I_2 - 4I_3 = 0$ ----------------------------(3)

Solving, $I_1 = 0.46\text{A}$, $I_2 = -0.923\text{A}$, $I_3 = -0.115\text{A}$

MODULE-3

Transient Behavior and Initial Conditions

By
K PRABHAVATHI
Assistant Professor
Department of ECE
BGSIT, BG NAGARA

Contents

- Behavior of circuit elements under switching condition and their Representation

- Evaluation of initial and final conditions in RL, RC and RLC circuits for AC and DC excitations.

Transient Response

When a network is switched from one condition to another by change in applied voltage or by change in one of the circuit elements, during the period of time, the branch currents and voltages change from their former values to new one. This time interval is called <u>transient period</u>.

The response or output of network during transition period is called <u>transient response of network.</u>

Initial Conditions

Initial Conditions in Elements

- **While solving problems on networks, using integro-differential equations in mathematics, initial conditions are always given and the job of the mathematician is to find the solution using the given initial conditions.**

- **Knowing the values of the voltages and currents of the elements at $t = 0^-$, finding these values at $t = 0^+$, constitutes the evaluation of initial conditions.**

Initial conditions of the elements in the network must be known to evaluate arbitrary constants in the general solution of differential equation. In the analysis of network behavior of elements individually and in combinations is studied with initial conditions.

Initial Conditions in Elements

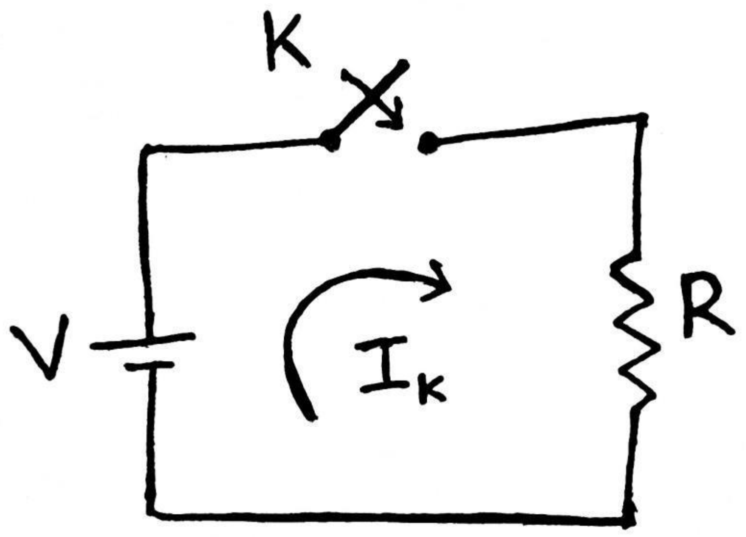
1. The Resistor

- When a voltage V is applied across the resistance R by closing the switch K, the current through R is given by

$$i_R = \frac{V}{R}$$

This equation indicates that the current through the resistor R changes instantaneously . Hence, in a resistor , the current changes instantaneously and the energy is dissipated as heat and it does not store any energy.

$$t = 0^-$$



Initial Conditions in Elements

1. The Inductor

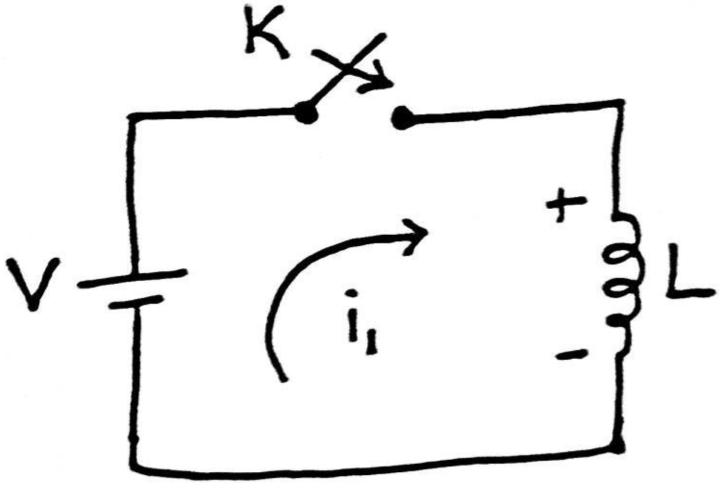
- When a voltage V is applied across the an inductor of inductance L henrys, the voltage across the inductance is given by,

$$v_L = L\frac{di_L}{dt}$$

If the current flowing through the inductor is DC, then $\frac{di_L}{dt} = 0$. Hence the voltage across the inductor is zero. Hence , under steady state conditions, the inductor acts as a short circuit.

The current through the inductance is given by:

$$\boldsymbol{i_L = \frac{1}{L}\int_{-\infty}^{t} v_L\,dt = \frac{1}{L}\int_{-\infty}^{0-} v_L\,dt + \frac{1}{L}\int_{0-}^{t} v_L\,dt}$$ **.....eq(1)**



The first term in the RHS of above equation represents the initial value of Current through the inductor before closing the switch i,e i_L(0-).

Initial Conditions in Elements

- When the switch is closed at t=0, the equation (1) can be written as

$$\boldsymbol{i_L(0+) = i_L(0-) + \frac{1}{L}\int_{0-}^{0+} v_L \, dt} \; \ldots\ldots\ldots \; \textbf{eq(2)}$$

It is assumed that the switching operation does not consume any time. Thus the integration from 0- to 0+ is zero.

$$i_L(0+) = i_L(0-)$$

Thus the current through an inductor cannot change instantaneously. This means that the current through the inductor before and after a switching operation is the same.

Hence, at t= 0+, the inductor acts as an O.C, if it does not carry any initial current. If the inductor carries an initial current I_0 before a switching operation, then immediately after the switching operation i.e. at t= 0+ , it acts as a current source of I_0.

Initial Conditions in Elements

1. The capacitor

- The current through the capacitance is given by,

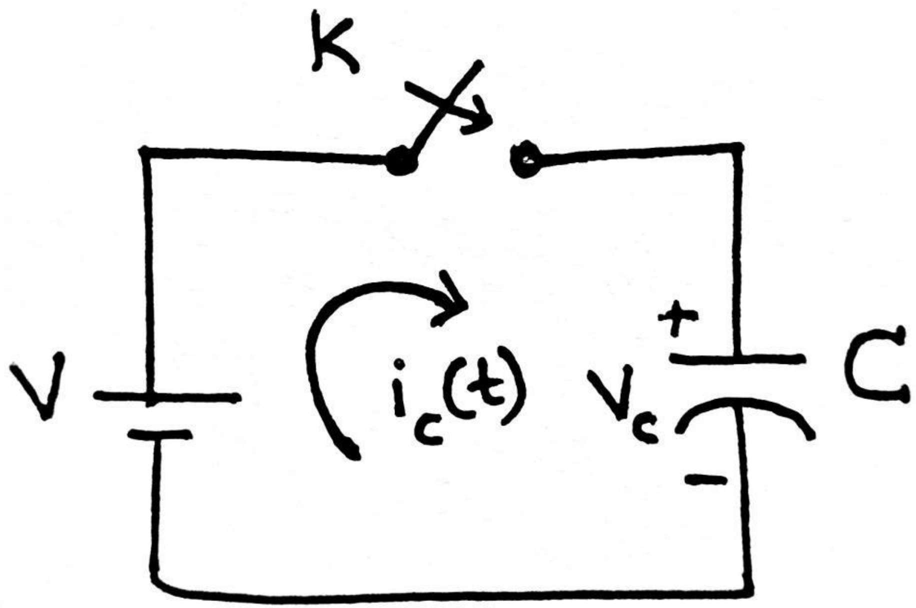
$$i_C = C\frac{dv_C}{dt}$$

If a DC voltage is applied, $\frac{v_C}{dt}$ = 0 and hence i_C = 0. Hence , under steady state conditions, capacitance acts as an O.C. Hence, under steady state conditions , the capacitance acts as an O.C.

The voltage across the capacitance is given by:

$$v_C = \frac{1}{C}\int_{-\infty}^{t} i_C\, dt = \frac{1}{C}\int_{-\infty}^{0-} i_C\, dt + \frac{1}{C}\int_{0-}^{t} i_C\, dt \ldots\ldots eq(3)$$

$$\frac{1}{C}\int_{-\infty}^{0-} i_C dt = v_C(0-), \textit{which is constant}.$$



Initial Conditions in Elements

- When the switch K is closed at t= 0, the equation (3) may be written as

$$v_C(0+) = v_C(0-) + \frac{1}{C}\int_{0-}^{0+} i_C\, dt = v_C(0-) + 0$$

Therefore

$$v_C(0+) = v_C(0-)$$

Thus the voltage across the capacitor can not change instantaneously.

Hence, if a capacitor does not have any initial charge at t= 0- , then at t=0+ also, its voltage will be zero.

Thus , the capacitor acts as S.C. at t=0+. If at t=0-, the capacitor has an initial voltage of V_0, then at t=0+ , it acts as a voltage source of V_0.

Procedure for finding the initial conditions

1. The initial values of voltages and currents i.e. before closing the switch at t= 0- , can be found directly from the schematic diagram of the given network.
2. For each element of the network, we must find out , what happens to the element at t= 0+ , i.e. after closing the switch.
3. A new equivalent network t=0+ is constructed as per the following rules:

- Replace all the inductors by open circuits or current sources having values of current flowing at t= 0-.
- Replace all the capacitors by short circuits or voltage sources of q0/C , if there is any initial charge.
- Resistors are left in the network without any change.

4. From the network at t=0+, first the initial values of voltages and current are solved.
5. After closing the switch, the general equation of the circuit is written.
6. By substituting the initial conditions, the initial condition of the first order derivative is found.
7. The general equation is differentiated again and by substituting the initial conditions, the initial conditions of the second order derivative can be found.

Behaviour of the circuit elements under switching conditions

t= 0 is the reference time when switch is operated , to differentiate time immediately before and after switching we use t=0- and t=0+.

The time at final condition is considered as at t=∞

Note*

$$Q = CV$$

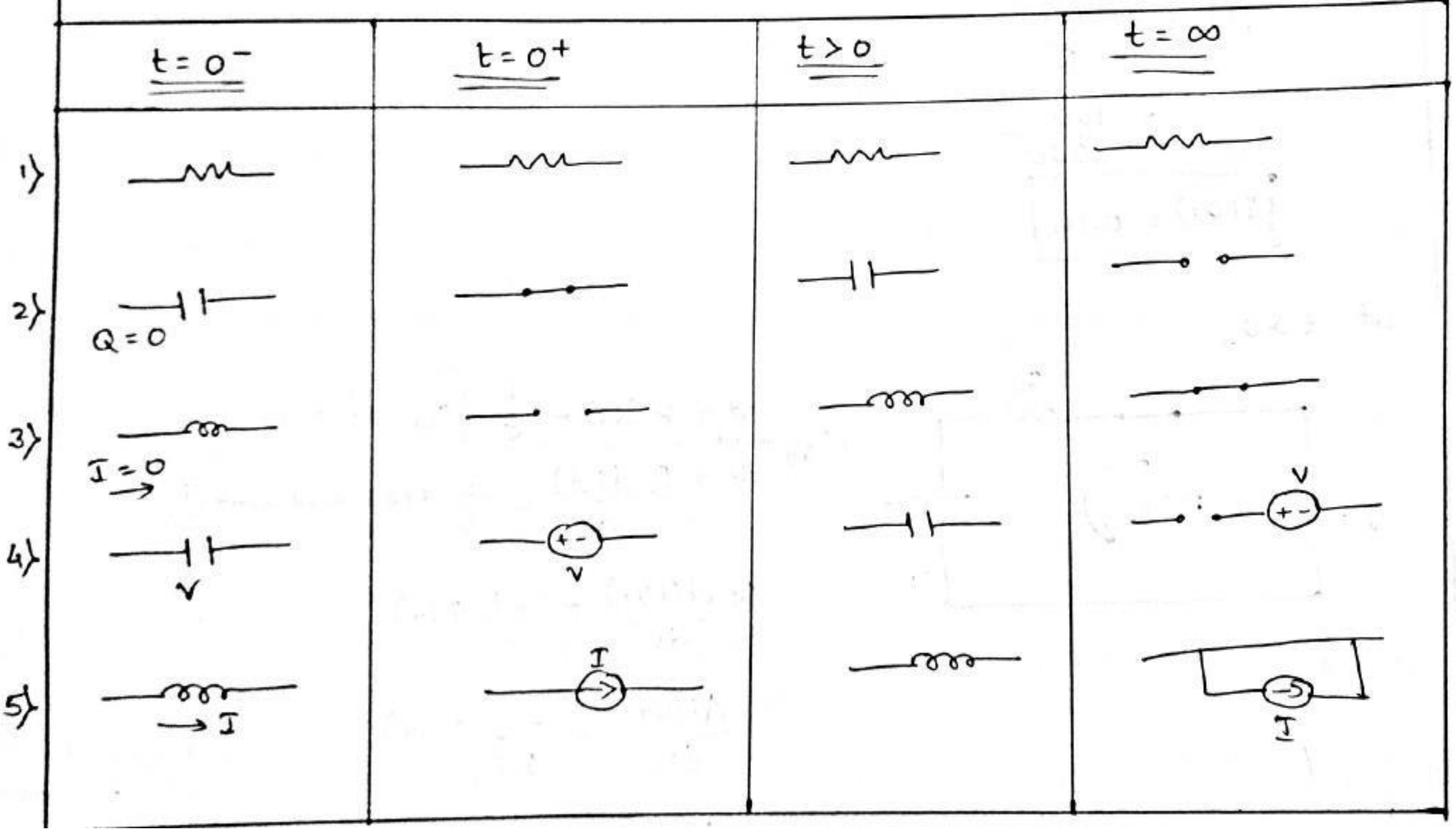
$$\frac{dQ}{dt} = C\frac{dv}{dt}$$

$$i = C\frac{dv}{dt}$$

$$\boxed{v = \frac{1}{C}\int i\,dt}$$

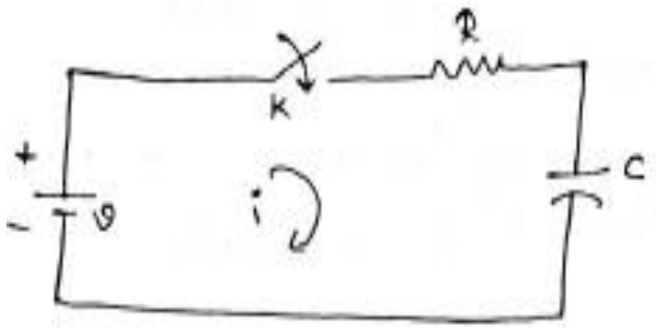
$$v_L = L\frac{di}{dt}$$

$$\boxed{i_L = \frac{1}{L}\int V_L\,dt.}$$



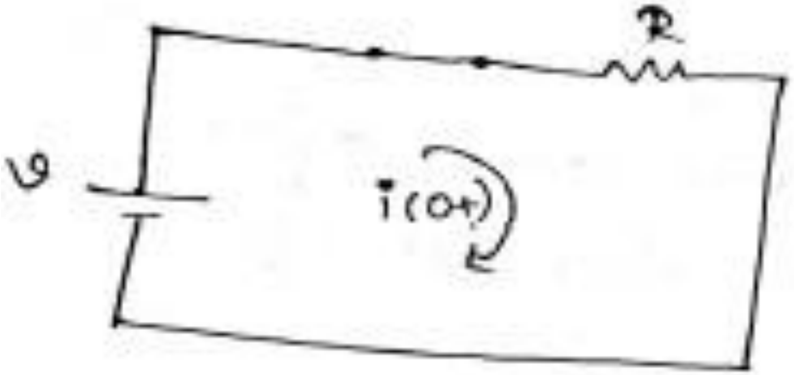
t = 0⁻
t = 0⁺
t > 0
t = ∞
1)
2)
Q = 0
3)
I = 0
4)
V
V
V
5)
I
I
I

Problem1: **In the network of figure, the switch K is closed at t=0,with the capacitor uncharged. Find values for i,** $\frac{di}{t}$ **&** $\frac{di^2}{dt^2}$ **at t=0+ for element values as follows v=100v, R=1000ohms and c=1µF.**



Solution :

At t=0+



$$i(t) = \frac{v}{R}$$

$$i(0+) = \frac{v}{R}$$

$$= \frac{100}{1000}$$

$$\boxed{i(0+) = 0.1A}$$

$$v - Ri(t) - \frac{1}{c}\int (t)\mathrm{dt}=0$$

Diff w.r.t ‘t’

$$0 - \frac{Rdi_{(t)}}{dt} - \frac{1}{c} i(t) = 0 \ldots\ldots e(1)$$

$$\frac{Rdi(0+)}{dt} = -\frac{1}{c} i(t)$$

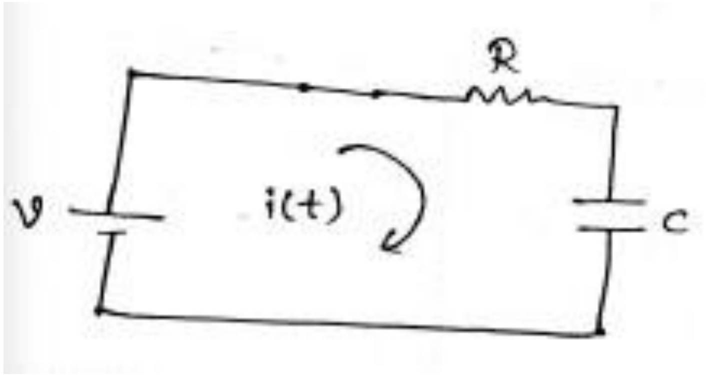
$$\frac{di(0+)}{dt} = -\frac{1}{Rc} i(0+)$$

$$\frac{di(0+)}{dt} = -1000\, i(0+)$$

$$\frac{di(0+)}{dt} = -1000\ X\ 0.1$$

$$\boxed{\frac{di(0+)}{dt} = -100\ A/Sec}$$

At t > 0



Diff equation 1 w. r.t ‘t’

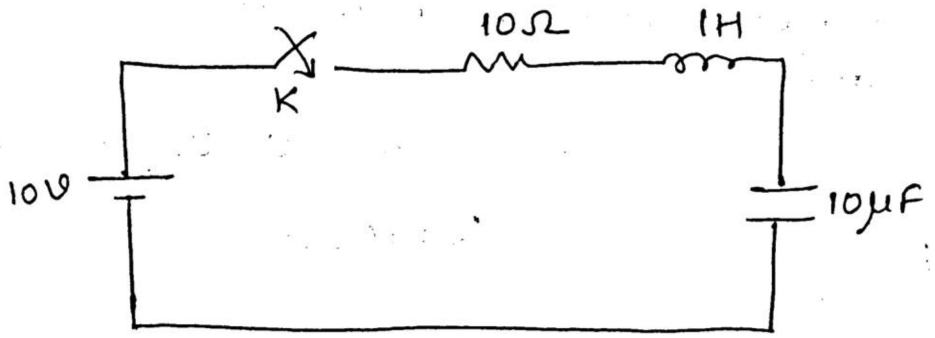
$$-R\frac{d^2 i_{(t)}}{dt^2} - \frac{1}{c}\frac{di_{(t)}}{dt} = 0$$

$$\frac{d^2 i(0+)}{dt^2} = -\frac{1}{Rc}\frac{di(0+)}{dt}$$

= -1000 X (-100)

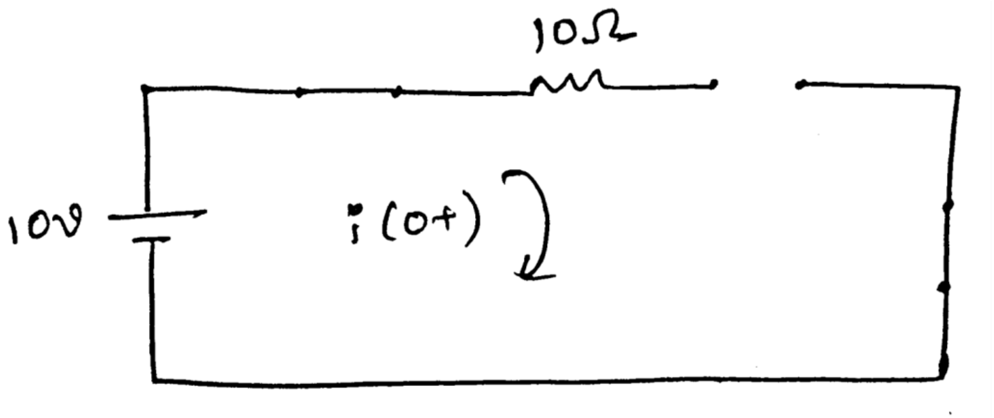
$$\boxed{\frac{d^2 i(0+)}{dt^2} = 100000\ A/sec^2}$$

Problem 2: **In the circuit given, the voltage across capacitor is previously uncharged. Find i(0+) , $\frac{di(0+)}{dt}$ & $\frac{d^2i(0+)}{dt^2}$. The switch K is closed at t=0.**



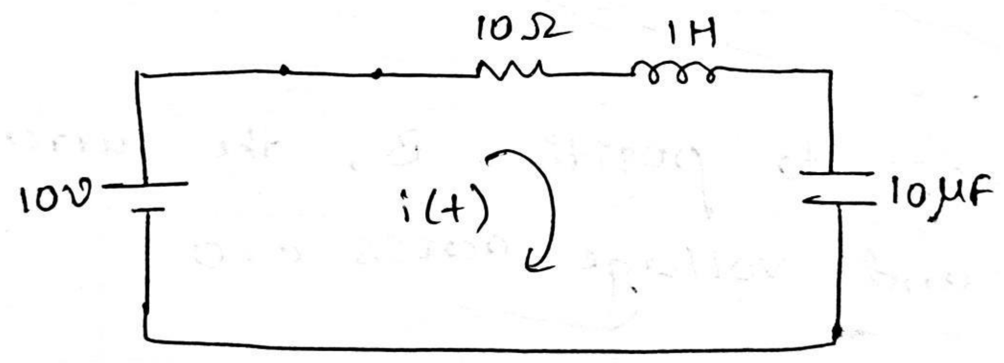
Solution :

At t=0+



$$i(0+) = 0\ A$$

At t > 0



$$10 - 10\ (t) - L\frac{di(t)}{dt} - \frac{1}{c}\int i(t)dt = 0 \ldots e(1)$$

$$10 - 10i(0^{+}) - L\frac{di(0^{+})}{dt} - \frac{1}{c}\int i(0^{+})dt = 0$$

$$10 - 0 - L\frac{di(0^{+})}{dt} - 0 = 0$$

$$10 - L\frac{di(0+)}{dt} = 0$$

$$\frac{di(0+)}{dt} = \frac{10}{1} = 10\ A/Sec$$

Diff equation (1) w. r.t 't'

$$0-\frac{10di(t)}{dt}-L\frac{d^2i(t)}{dt^2}-\frac{1}{c}i(t)dt=0$$

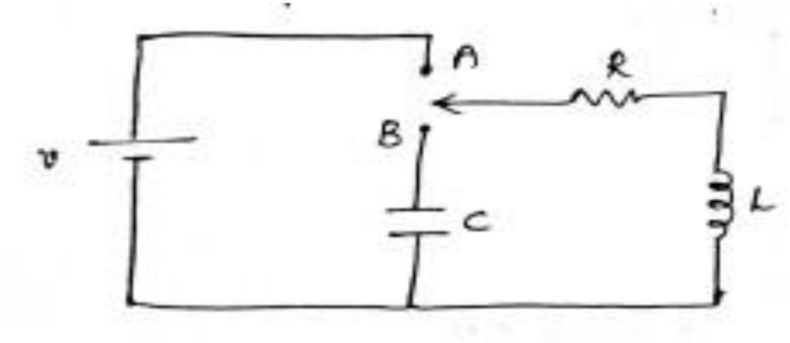
$$-\frac{10di(0^+)}{dt}-L\frac{d^2i(0^+)}{dt^2}-\frac{1}{c}i(0^+)dt=0$$

$$L\frac{d^2i(0^+)}{dt^2}=-\frac{10di(0^+)}{dt}$$

$$L\frac{d^2i(0^+)}{dt^2}=-10\ X\ 10=-100$$

$$\frac{d^2i(0^+)}{dt^2}=-100\ A/sec^2$$

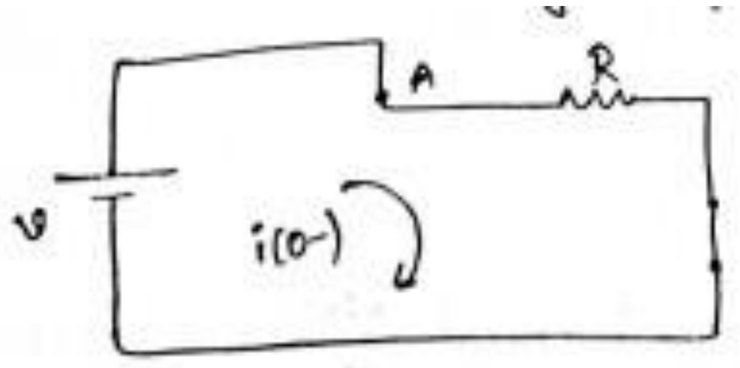
Problem 3 : **In the network given , K is changed from position A to B. Solve for i(0+),** $\frac{di(0+)}{dt}$ & $\frac{d^2i(0+)}{dt^2}$ **, circuit reached steady stat(t = ∞) in position A.**



V= 100 v , c = 0.1$\mu F, L = 1\ H\ , R = 1000\ ohms$

Solution :

$At\ t = 0^{-}\ steady\ state\ at\ A$

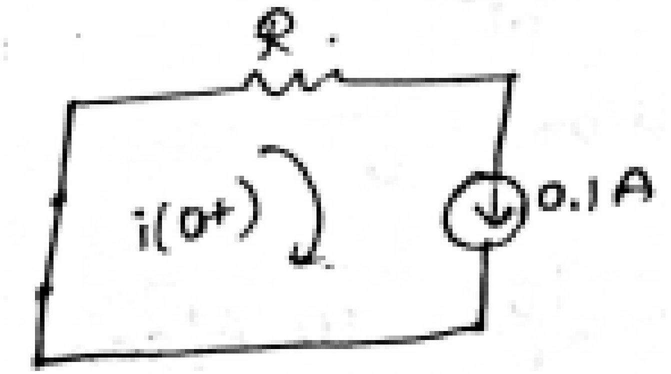


$i(0^-) = \frac{100}{1000}$

$i(0^-) = 0.1A$

Prior to the switch change to position B, the current through inductor is 0.1 A and voltage across c = 0

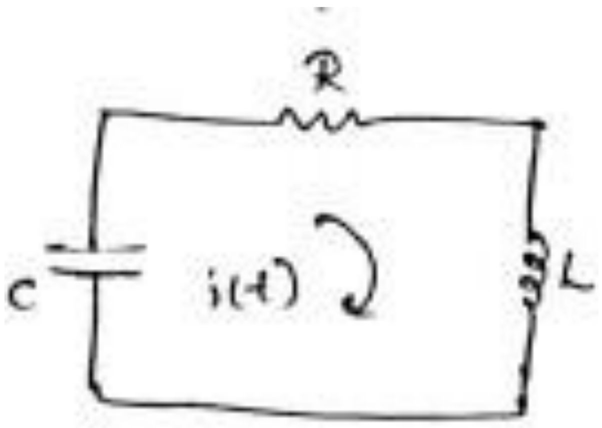
At t = 0+



$$i(0^+) = 0.1\ A$$

At t > 0

$$-\frac{1}{c}\int i(t)dt - R\ (t) - L\frac{di_{(t)}}{dt} = 0$$



$$-\frac{1}{c}\int i(0^+)dt - R\ (0^+) - L\frac{di(0^+)}{dt} = 0 \ldots\ldots e(1)$$

$$0 - 0.1(1000) - 1\,\frac{di(0^+)}{dt} = 0$$

$$\frac{di(0^+)}{dt} = -100\ \text{A/Sec}$$

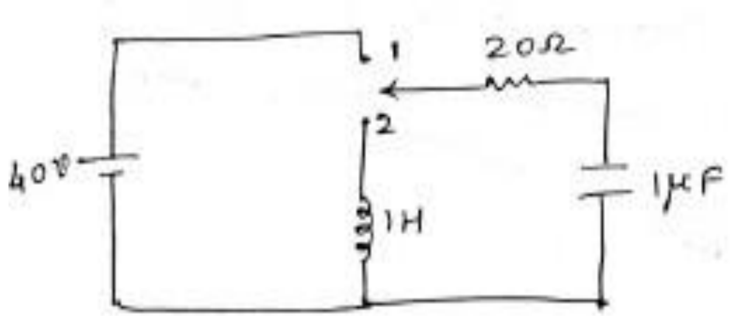
Diff equation (1) w. r.t ‘t’

$$-\frac{1}{c}i(0^+)-\frac{Rdi(0^+)}{dt}-L\frac{d^2i(0^+)}{dt^2}=0$$

$$-\frac{1}{0.1\mu}(0.1)-1000(-100)=L\frac{d^2i(0^+)}{dt^2}$$

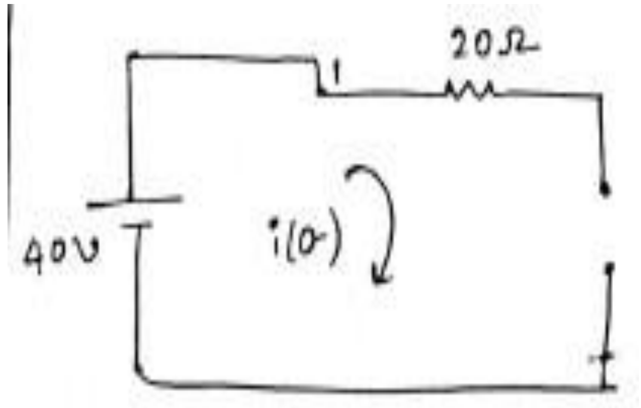
$$\frac{d^2i(0^+)}{dt^2}=-9\ X\ 10^5\ A/sec^2$$

Problem 4 : **In the network shown , switch is moved from position 1 to 2 at t = 0. The steady state have been reached before switching. Calculate i(0+), $\frac{di(0+)}{dt}$ & $\frac{d^2i(0+)}{dt^2}$.**



Solution :

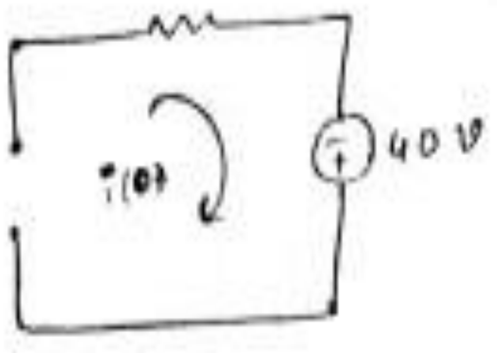
$$At\ t = 0^-$$



$$i(0^-) = 0$$

$$V_c = 40\ v$$

$At\ t = 0^{+}$



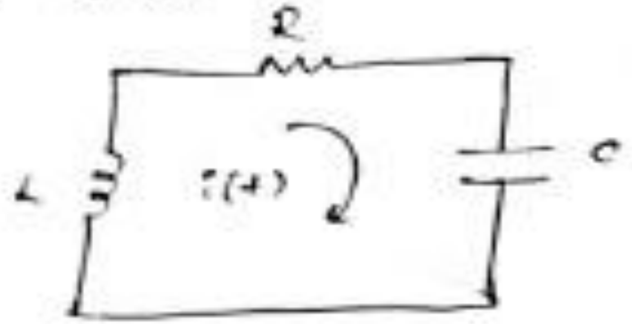
$$i(0+) = 0$$

$$-\frac{1}{c}\int i(t)dt - Ri(t) - L\frac{di(t)}{dt} = 0$$

$$-40 - 20(0) - 1\frac{di(t)}{dt} = 0$$

$$\frac{di(t)}{dt} = -40\ A/Sec$$

$At\ t > 0$



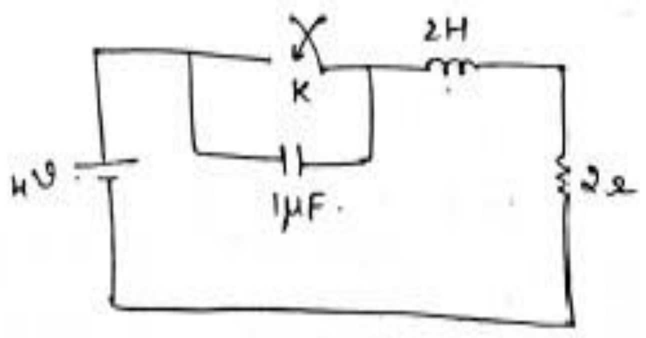
Diff equation (1) w. r.t 't'

$$-\cancelto{0}{\frac{1}{c}i(t)} - R\frac{di(t)}{dt} - \frac{Ld^2i(t)}{dt^2} = 0$$

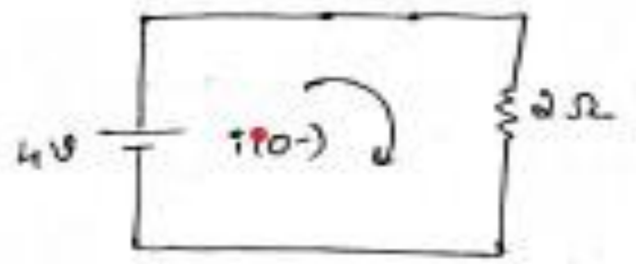
$$-20(-40) = 1.\frac{d^2i(t)}{dt^2}$$

$$\frac{d^2i(0^+)}{dt^2} = 800\ A/sec^2$$

Problem 5 : **In the network given ,steady state is reached with switch is closed at t= 0,switch is open. Determine i(0+), $\frac{di(0+)}{dt}$, $v_k(0^+)$, $\frac{dv_k(0^+)}{dt}$.**



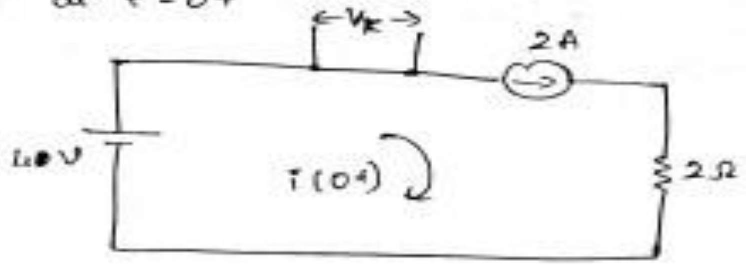
at t = 0⁻



$i(0^-) = 2A$

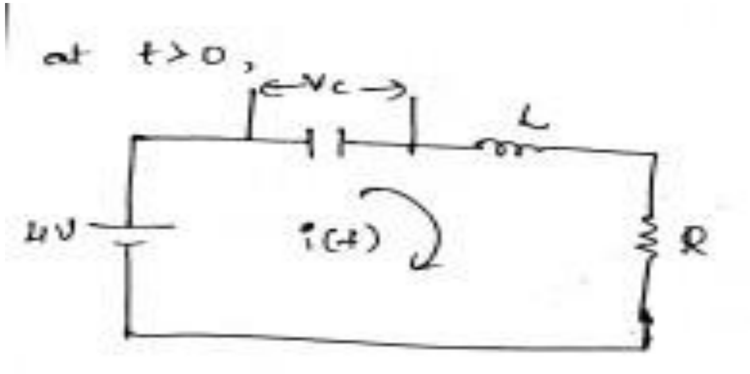
$V_c = 0\ v$

at t = 0+



$i(0^+) = 2A$

$V_k(0^+) = 0v = V(0^+)$



$$4-\frac{1}{c}\int i(0^+)dt - Ri(0^+) - L\frac{di(0^+)}{dt} = 0$$

$$-4-0-2*2-L\frac{di(0^+)}{dt} = 0$$

$$4-4-2\frac{di(0^+)}{dt} = 0$$

$$\frac{di(0^+)}{dt} = 0\ A/Sec$$

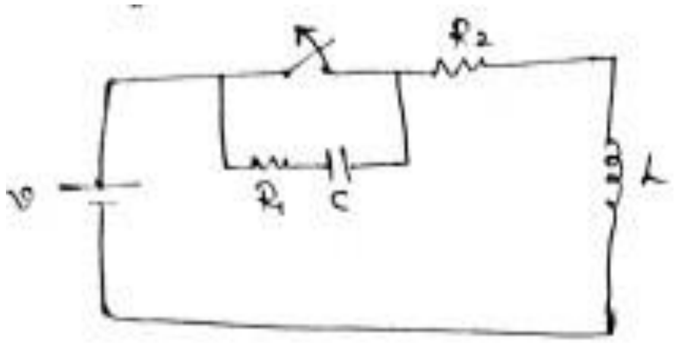
$$V_c = \frac{1}{c}\int i\ dt$$

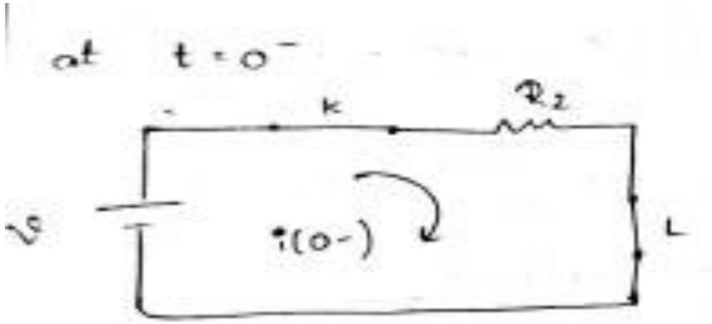
$$\frac{dv}{dt} = \frac{1}{c}i(0^+)$$

$$\frac{dv_k}{dt} = \frac{2}{1*10^{-6}}$$

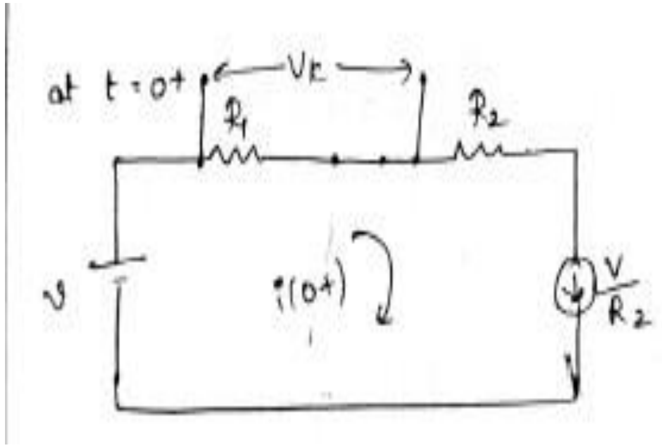
$$= 2*10^6 v/sec$$

Problem 6 : In the given network , switch is opened at t=0, after the network has attained the steady state with switch closed. Find the expression for voltage across the switch at t=0+, if the parameters are such that $i(0^+) = 1A$ and $\frac{di(0^+)}{dt} = -\frac{1A}{sec}$. What is the value of derivative of the voltage across the switch.

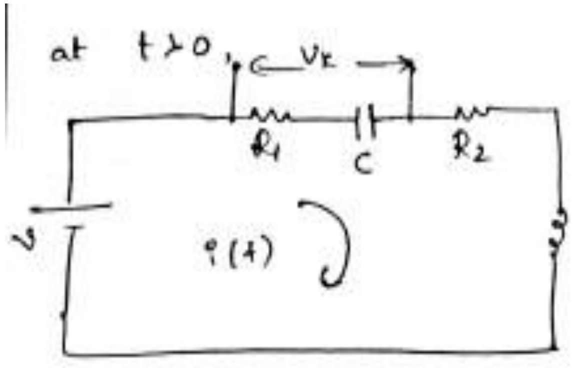




$$i(0^-) = \frac{V}{R_2}$$
$$V_c = 0$$



$$i(0^+) = \frac{V}{R_2}$$
$$v_k(0^+) = i(0^+) * R_1$$
$$v_k(0^+) = \frac{V}{R_2} * R_1$$



$$V_k = V_{R_1} + V_c$$

$$V_k = i(t)R_1 + \frac{1}{c}\int i(t)dt$$

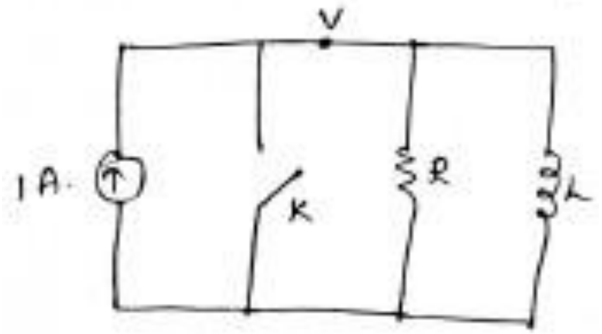
$$\frac{dV_k}{dt} = \frac{di(t)}{dt}R_1 + \frac{1}{c}i(t)$$

$$\frac{dV_k}{dt} = -R_1 + \frac{1}{c}(1)$$

$$\frac{dV_k}{dt} = \frac{1}{c} - R_1$$

Problem 7 : **In the circuit given , switch k is open at t= 0, Determine** $v(0^+), \frac{dv(0^+)}{dt}, \frac{d^2v(0^+)}{dt^2}$ *and*

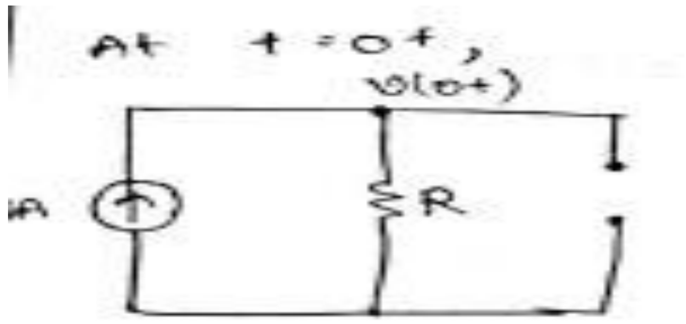
$$given\ \ R = 100\ ohm, L = 1H$$



Solution :

$$t = 0^-$$

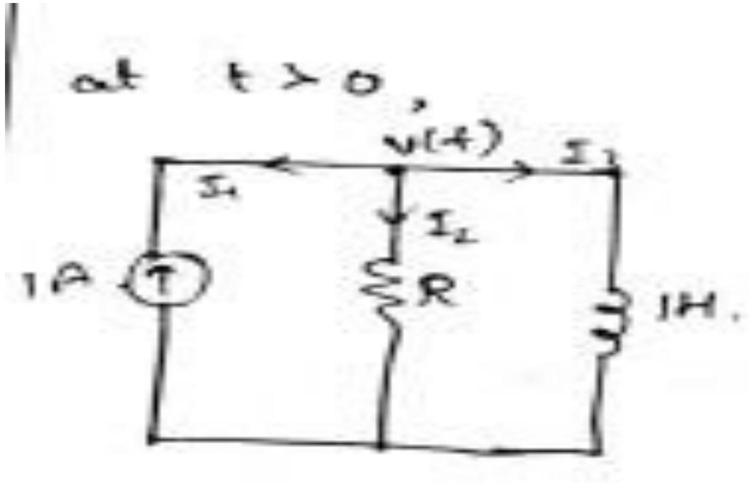
Since switch is closed no current flows in inductor $i_L = 0$.



$$v(0^+) = IR$$

= 1*100

=100 v



$$-I_1 - I_2 - I_3 = 0$$

$$-(-1) - \frac{v(t)}{R} - \frac{1}{L}\int v(t)dt = 0 \ldots .eq(1)$$

Diff equation (1) w.r.t 't'

$$0 - \frac{dv(t)}{dt}.\frac{1}{R} - \frac{1}{L}v(t) = 0 \ldots \ldots \ldots eq(2)$$

$$\frac{dv(t)}{dt}.\frac{1}{R} = -\frac{1}{L}v(t)$$

$$\frac{dv(t)}{dt} = -\frac{R}{L}v(t)$$

= -100(100)

$$\frac{dv(t)}{dt} = -10000 = -10^4\ v/sec$$

$Diff\ equation\ w.r.t\ 't'$

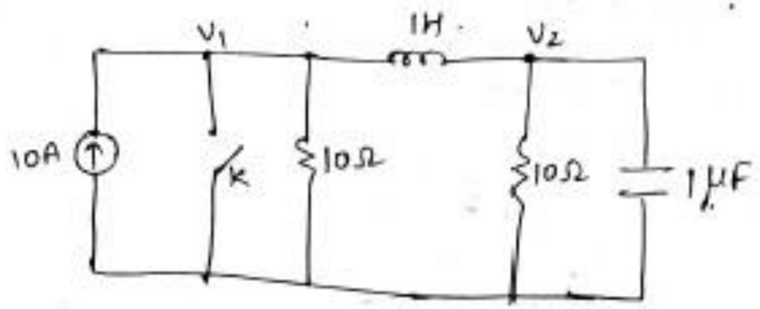
$$\frac{d^2v(t)}{dt^2}\cdot\frac{1}{R}+\frac{1}{L}\frac{dv(t)}{dt}=0$$

$$\frac{d^2v(t)}{dt^2}=-\frac{R}{L}(-10^4)$$

$$= -100(-10^4)$$

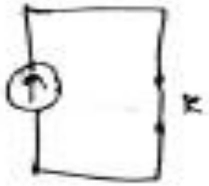
$$\frac{d^2v(t)}{dt^2}=10^6 v/\sec^2$$

Problem 8 : **In the circuit given, switch K is open at t=0. Find $v_1(0^+)$ and $v_2(0^+)$ and $\frac{dv_1(0^+)}{dt}$ and $\frac{dv_2(0^+)}{dt}$.**



Solution :

At $t=0^-$

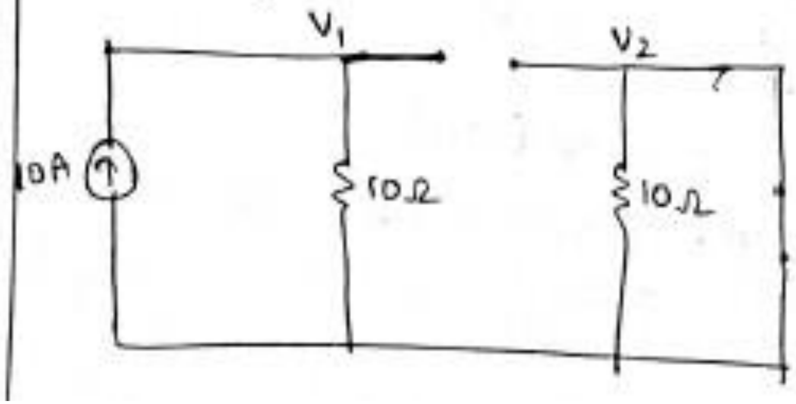


at $t=0^-$,

no current flows in capacitor

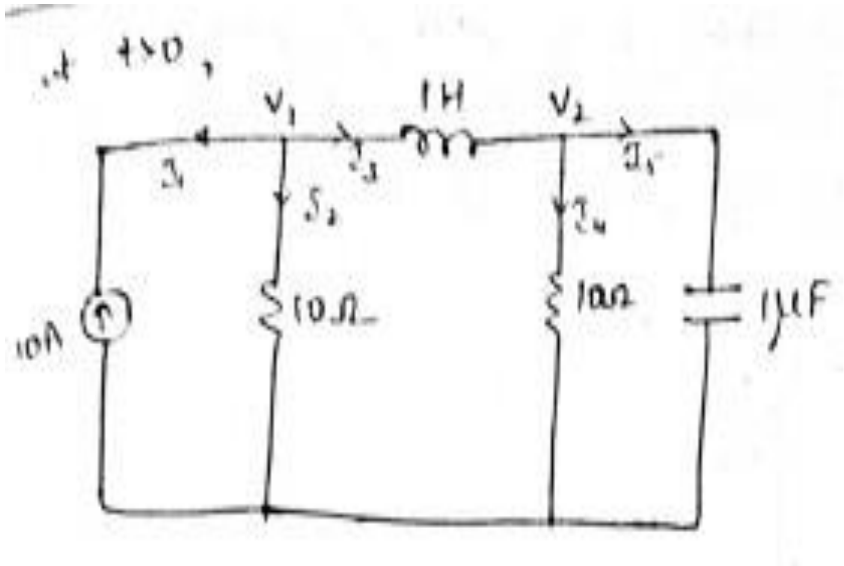
$V_1=0$ $V_2=0$

At $t=0^+$,



$$v_1(0^+) = 10 * 10 = 100v$$

$$v_2(0^+) = 0v$$



$$I_3 - I_4 - I_5 = 0$$

$$\int \left[v_1 - \frac{v_2}{L}\right] dt - \frac{v_2}{10} - c\frac{dv_2}{dt} = 0$$

0 0

$$\frac{1}{L}\int [v_1 - v_2]dt - \frac{v_2}{10} - c\frac{dv_2(t)}{dt} = 0$$

$$\frac{dv_2(t)}{dt} = 0$$

$$-I_1 - I_2 - I_3 = 0$$

$$-(-10) - \frac{v_1}{10} - \frac{1}{1}\left[v_1 - \frac{v_2}{L}\right] = 0$$

$$10 - \frac{v_1}{10} - \frac{1}{L}\int v_1 dt = 0$$

$$-10 + \frac{v_1(t)}{10} + \frac{1}{L}\int v_1 dt = 0$$ ....eq (1)

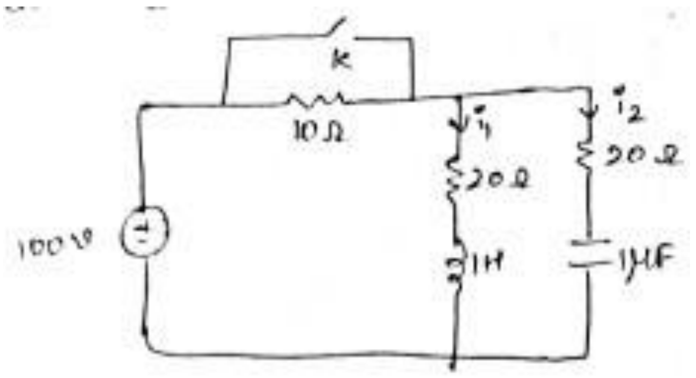
Diff Eq (1)

$$0 + \frac{dv_1(0^+)}{dt}\cdot\frac{1}{10} + 1.v_1(0^+) = 0$$

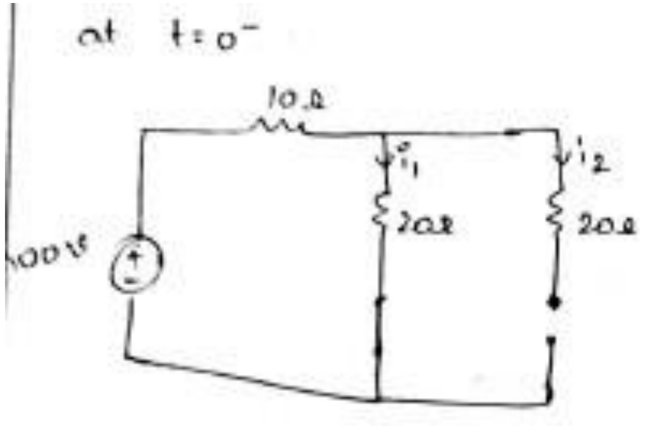
$$\frac{dv_1(0^+)}{dt}\cdot\frac{1}{10} = -100$$

$$\frac{dv_1(0^+)}{dt} = -100 * 10 = -1000\, v/sec$$

Problem 9: In the given network, switch K is open, steady state is reached at t=0 , switch is closed. Find $I_1, I_2, \frac{dI_1}{dt}, \frac{dI_2}{dt}$ at $t = 0^+$, also find $\frac{dI_1}{dt}$ at $t = \infty$.



Solution :

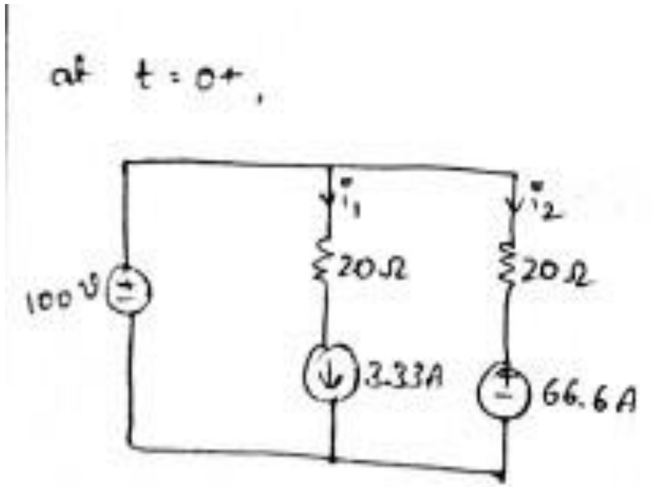


$$i_2 = 0A$$

$$i_1 = \frac{100}{30} = 3.33A$$

$$V_c = 20 * i = 20 * 3.33A$$

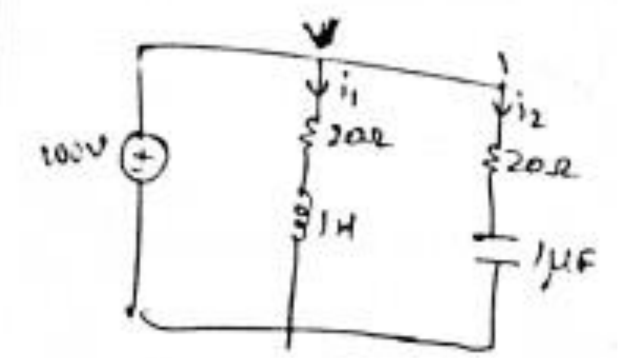
V_c =66.6 v



$$i(0^+) = 3.33A$$

$$i(0^+) = 100 - \frac{66.6}{20} = 1.67A$$

at $t>0$



$$100 = i_1(t)20 + \frac{Ldi_1(t)}{dt}$$

$$100 = i_1(o^+)20 + \frac{di_1(t)}{dt}$$

$$\frac{di_1(t)}{dt} = 33.4\ \mathrm{A/Sec}$$

$$100 = i_2(t)20 + \frac{1}{c}\int i_2(t)dt$$

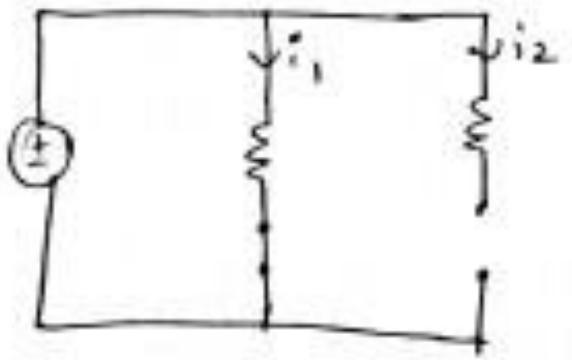
$$0 = \frac{di_2(t)}{dt}20 + \frac{1}{c}i_2(t)$$

$$\frac{di_2(t)}{dt}20 + \frac{1.67}{1\mu} = 0$$

$$\frac{di_2(0^+)}{dt}20 = -\frac{1.67}{1\mu}$$

$$\frac{di_2(0^+)}{dt} = -83.5 * 10^{-3}\ A/Sec$$

at $t = \infty$,



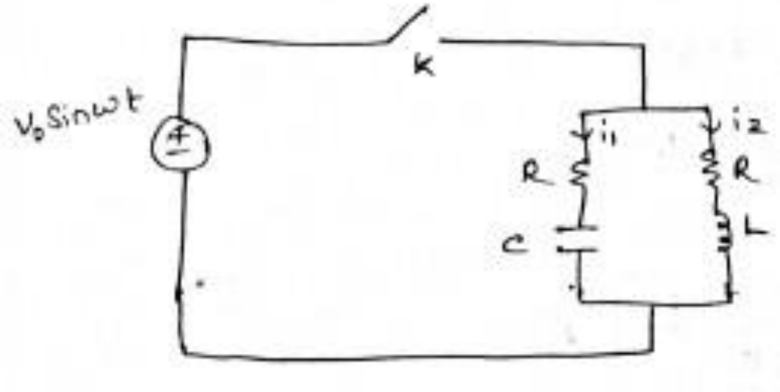
$$100 - 20i_1(\infty) = 0$$

$$i_1(\infty) = \frac{100}{20} = 5$$

$$0 - \frac{20di_1(\infty)}{dt} = 0$$

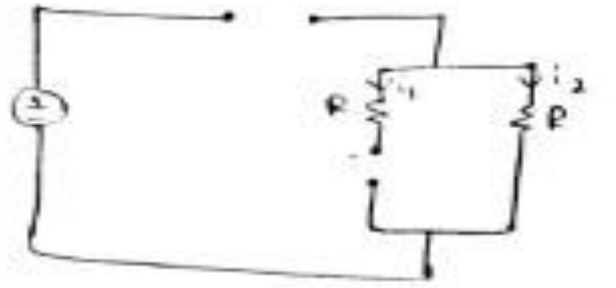
$$\frac{di_1(\infty)}{dt} = 0$$

Problem 10 : In the circuit given , switch K is closed at t=0,determine $i_1, i_2, \frac{di_1}{dt}, \frac{di_2}{dt}$ $at\ t = 0^+\ condition.$



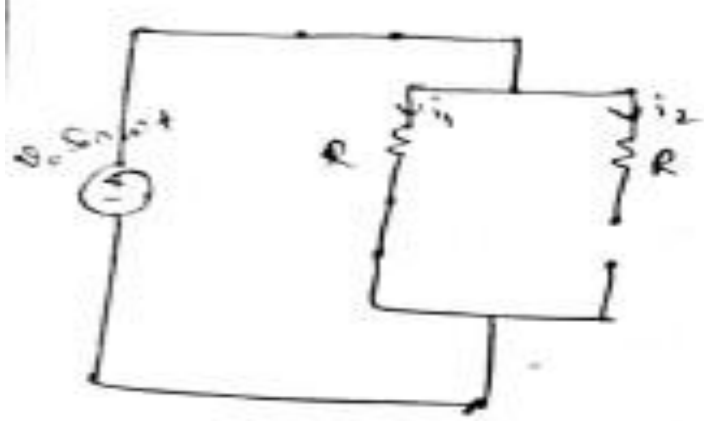
Solution :

at $t = 0^-$



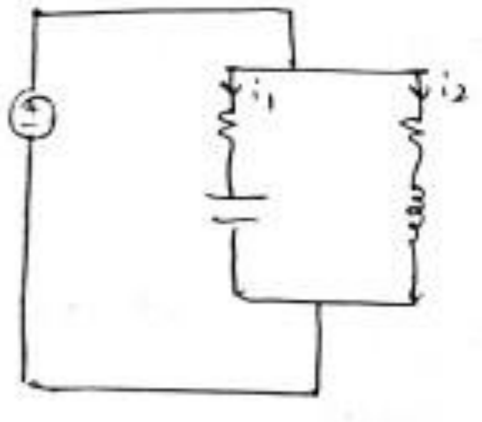
$$i_1 = 0$$
$$i_2 = 0$$
$$v_c = 0$$

at $t = 0^+$.



$$i_2(0^+) = 0$$

$$i_1(0^+) = \frac{v_0 Sinwt}{R}$$



$$V_0Sinwt = i_2(t)R + \frac{Ldi_2(t)}{dt}$$

$$V_0Sinwt = \overset{0}{i_2(0^+)}R + \frac{Ldi_2(t)}{dt}$$

$$\frac{di_2(t)}{dt} = \frac{V_0Sinwt}{L}$$

$$V_0Sinwt = i_1(t)R + \frac{1}{c}\int i_1(t)dt$$

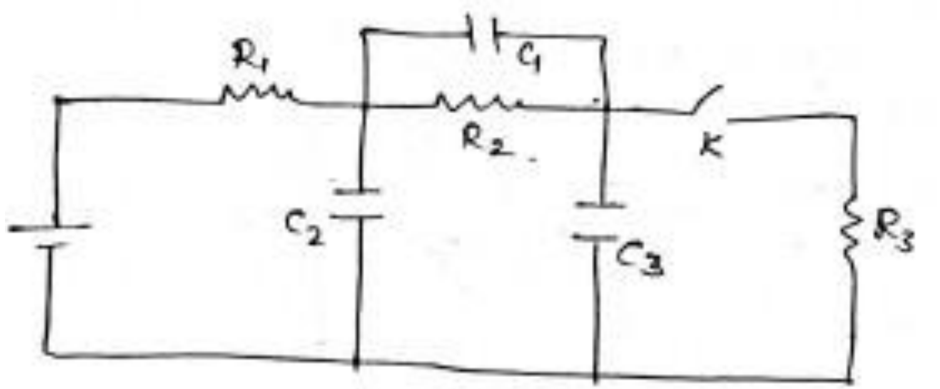
$$wV_0\ coswt = \frac{di_1(t)}{dt}.R + \frac{1}{c}i_1(0^+)$$

$$wV_0coswt = \frac{di_1(0^+)}{dt}.R + \frac{1}{c}i_1(0^+)$$

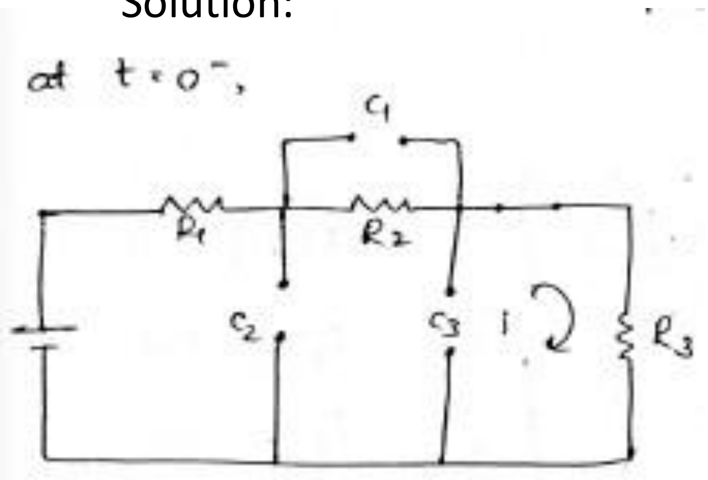
$$\frac{di_1(0^+)}{dt} = \left[wV_0coswt - \frac{V_0Sinwt}{\mathrm{RC}}\right].\frac{1}{\mathrm{R}}$$

$$\frac{di_1(0^+)}{dt} = \left[\frac{wV_0coswt}{R} - \frac{V_0Sinwt}{\mathrm{R}^2\mathrm{C}}\right].\frac{1}{\mathrm{R}}$$

Problem 11: **In the given network, switch K is open at t=0. Determine the voltage across the switch at $t = 0^{+}$, before steady state is reached.**



Solution:



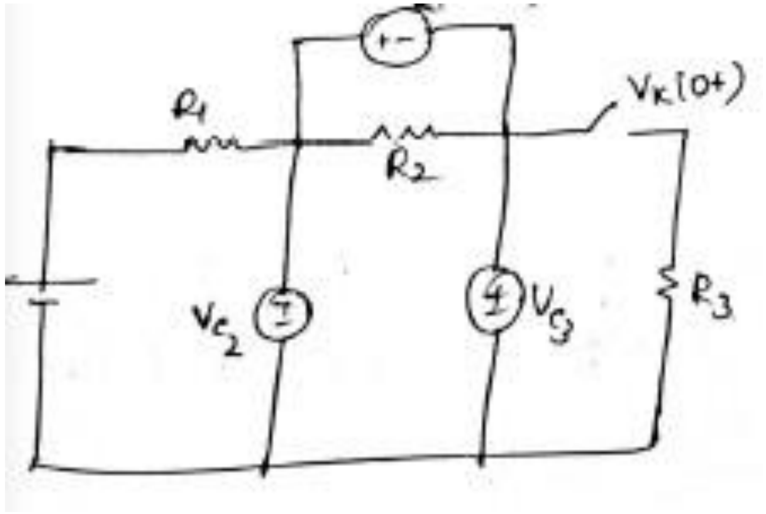
$$i = \frac{v}{R_1 + R_2 + R_3}$$

$$v_{c_1} = \left[\frac{v}{R_1 + R_2 + R_3}\right] . R_2$$

$$v_{c_2} = \left[v - \frac{vR_1}{R_1 + R_2 + R_3}\right]$$

$$v_{c_3} = \left[\frac{v}{R_1 + R_2 + R_3}\right] . \mathrm{R}_3$$

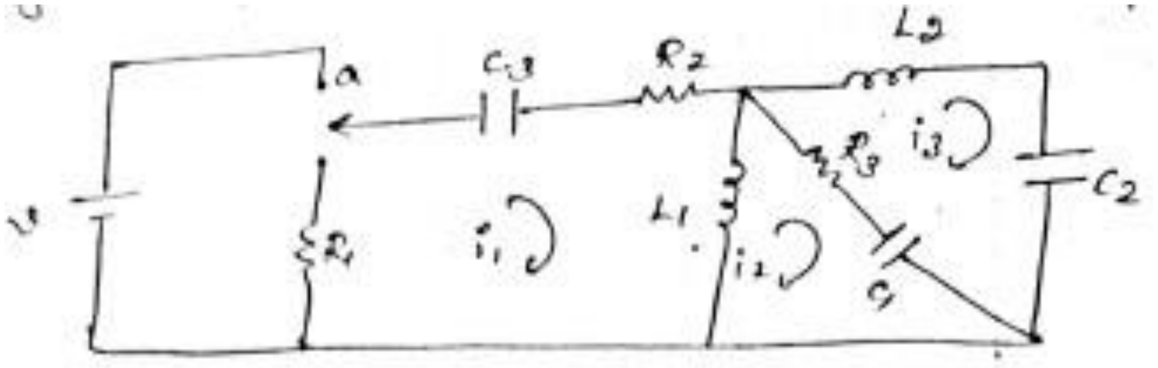
At $t = 0^+$



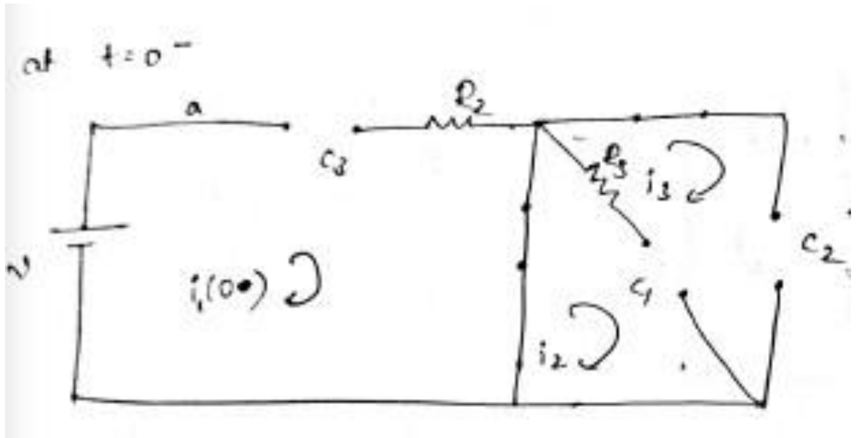
$$v_{c_1} = \left[\frac{v}{R_1 + R_2 + R_3}\right] . R_2$$

$$v_k(0^+) = \frac{vR_3}{R_1 + R_2 + R_3}$$

Problem 12: In the network given, the switch was in position A and steady state was reached at t=0, the switch is changed to position B. Determine $i_1(0^+)$, $i_2(0^+)$, $i_3(0^+)$



Solution :



$$i_1(0^-) = 0$$

$$i_2(0^-) = 0$$

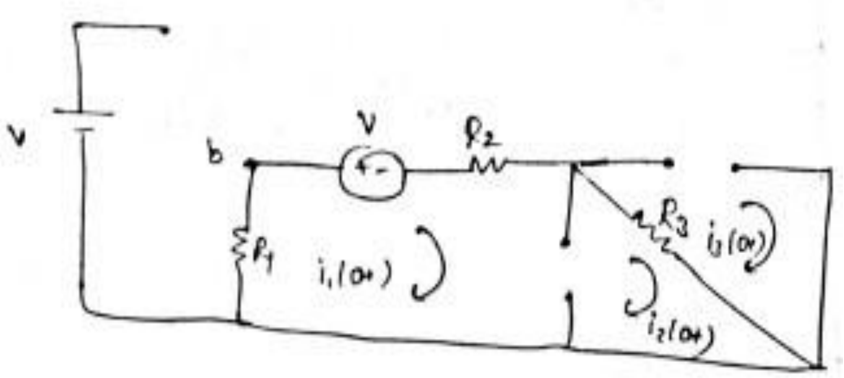
$$i_3(0^-) = 0$$

$$v_{c_1}(0^-) = 0$$

$$v_{c_2}(0^-) = 0$$

$$v_{c_3}(0^-) = v$$

at $t = 0+$,



$$i_3(0^+) = 0$$

$$i_2(0^+) = i_1(0^+)$$

$$i_2(0^+) = -\frac{v}{R_1 + R_2 + R_3} = i_1\ (0)$$

MODULE-4
Laplace Transformation & Applications

By
K PRABHAVATHI
Assistant Professor
Department of ECE
BGSIT, BG Nagara

Contents

- Solution of networks
- Step, ramp and impulse responses
- waveform Synthesis.

Laplace Transform

Laplace Transform

- The Laplace transform, named after its inventor Pierre-Simon Laplace is an integral transform that converts a function of a real variable t (often time) to a function of a complex variable s (complex frequency).
- It is very much suitable for obtaining solution of higher order differential equations.
- Mathematical Expression is given by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Standard Time Signals

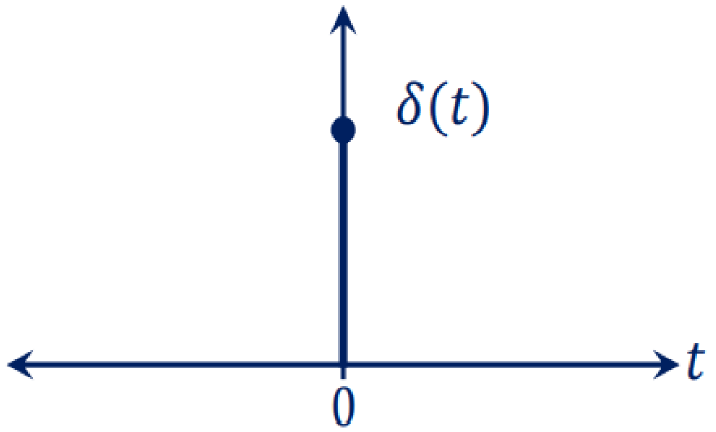
- Impulse function
- Step function
- Ramp function

Standard Time Signals

1. Impulse function

- Exists only at t=0 and is zero elsewhere.
- Also called as Dirac-delta function denoted by $\delta(t)$

$$\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$$



Laplace transform of Impulse function

$$L[\delta(t)] = \int_0^{\infty} \delta(t) e^{-st}\, dt$$

$$L[\delta(t=0)] = 1 * e^{-st}\Big|_{t=0}$$

$$L[\delta(t)] = e^{-s\,0}$$

$$\boldsymbol{L[\delta(t)] = 1}$$

Note: Differentiation of unit step function results in impulse function

$$\delta(t) = \frac{d}{dt}[u(t)]$$

$$L[\delta(t)] = L\left[\frac{d}{dt}[u(t))\right]$$

IIly

$$L\left[\frac{d}{dt} u(t)\right] = sL[u(t)] - u(t)\Big|_{t=0^-}$$

$$= \cancel{s} \times \frac{1}{\cancel{s}} - 0$$

$$L\left[\frac{d}{dt} u(t)\right] = 1$$

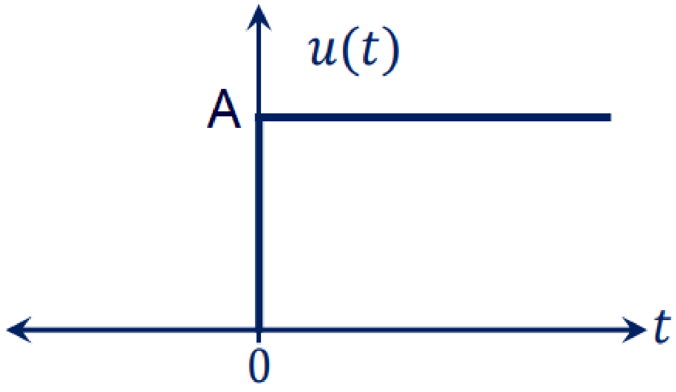
$$\left\langle L[\delta(t)] = 1 \right\rangle$$

Standard Time Signals

2. Step function

- Value of the function change from one value to another.
- The change may take place at t=0 or any other time.
- It is denoted by u(t).

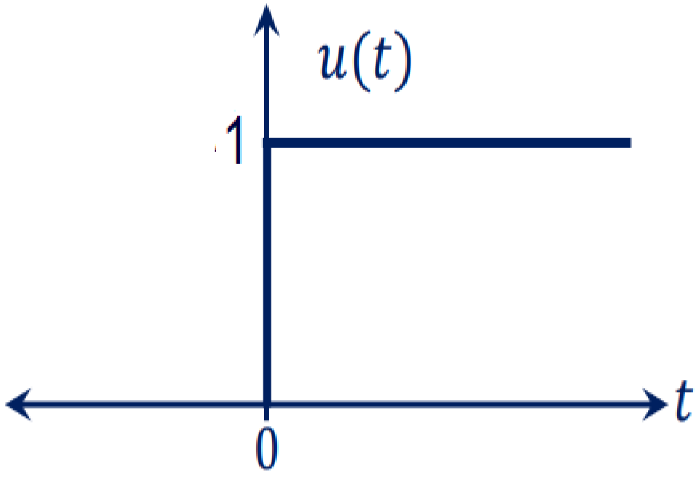
$$u(t) = \begin{cases} 0, & t < 0 \\ A, & t \geq 0 \end{cases}$$



Unit Step function

- Is one whose magnitude is equal to 1

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



Standard Time Signals

Laplace transform of Step function

$$L[u(t)] = \int_0^\infty u(t)e^{-st}\,dt$$

$$L[u(t)] = \frac{e^{-st}}{-s}\Big|_0^\infty$$

$$L[u(t)] = -\frac{1}{s}[0-1]$$

$$\boldsymbol{L[u(t)] = \frac{1}{s}}$$

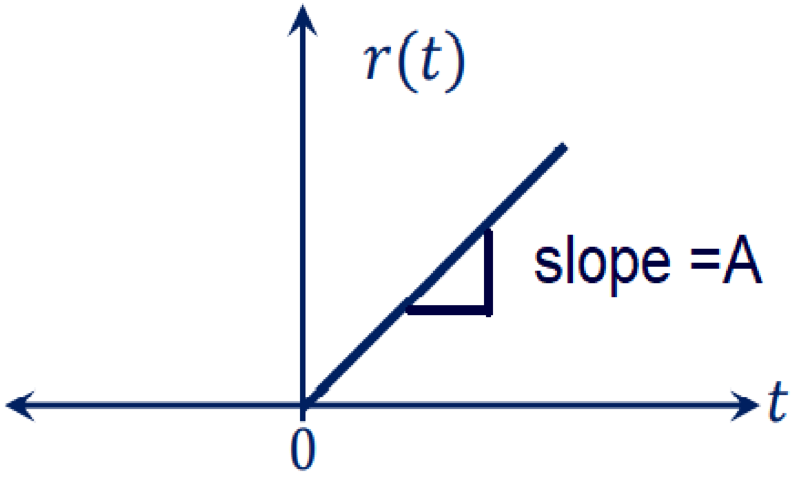
Standard Time Signals

3. Ramp function

- A ramp signal has a slop value for $t \geq 0$, otherwise it has zero value.
- The change may take place at t=0 or any other time.
- It is denoted by r(t).

$$r(t) = \begin{cases} 0, & t < 0 \\ At, & t \geq 0 \end{cases}$$

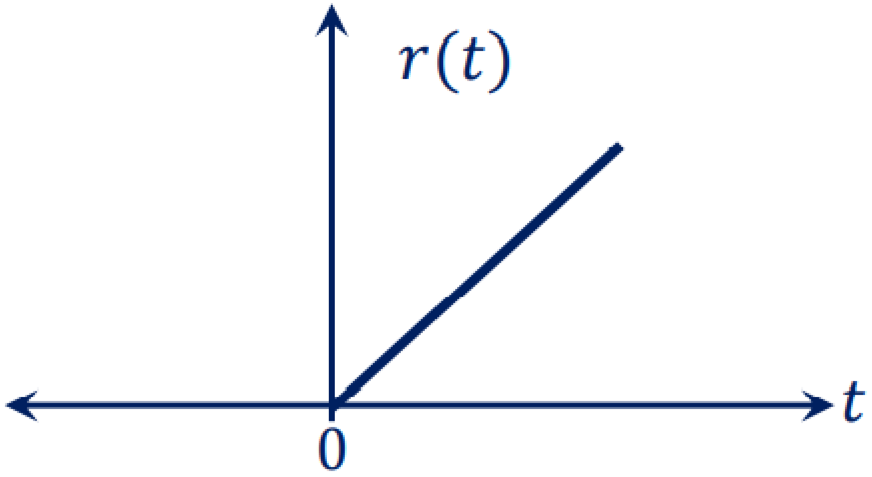
Where A is slope of the function



Unit Ramp function

- Is one which has unity slope value for $t \geq 0$

$$r(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$$



Standard Time Signals

Laplace transform of Ramp function

$$L[r(t)] = \int_0^\infty t\, u(t)\, e^{-st}\, dt$$

$$L[r(t)] = \int_0^\infty t * 1 * e^{-st}\, dt$$

$$L[r(t)] = t \frac{e^{-st}}{-s}\Bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} * 1\, dt$$

$$L[r(t)] = \frac{1}{s}(0-0) - \frac{e^{-st}}{s^2}\Bigg|_0^\infty$$

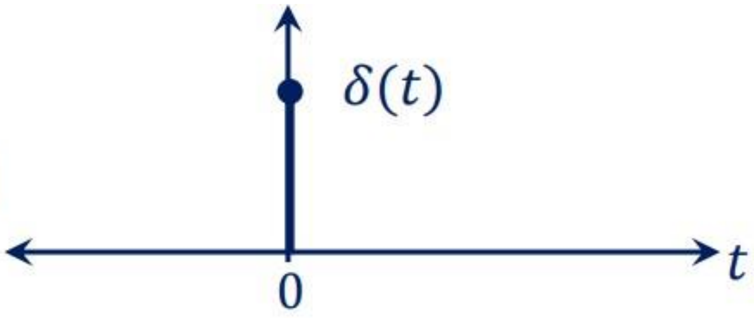
$$L[r(t)] = -\frac{1}{s^2}(0-1)$$

$$\boldsymbol{L[r(t)] = \frac{1}{s^2}}$$

Relations between Standard Time Signals

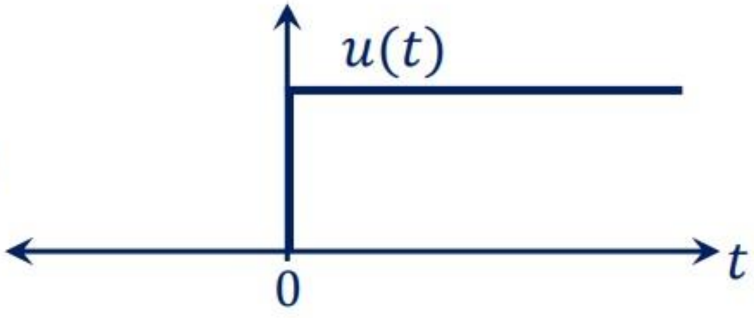
Integration

$\delta(t)$



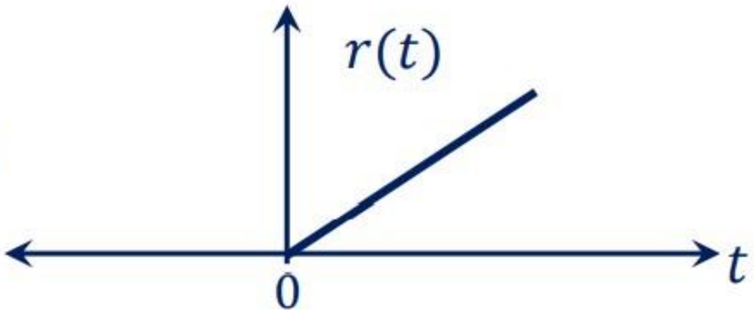
$$\frac{du(t)}{dt} = \delta(t)$$

$$\int \delta(t)dt = u(t)$$



$$\frac{dr(t)}{dt} = u(t)$$

$$\int u(t)dt = t = r(t)$$



$$r(t) = t$$

Differentiation

Shifting Theorem

Shifting Theorem

- Used to obtain Laplace transform of shifted functions.
- Consider a function f(t). Let this function be delayed by 'T' time units.
- It can be represented as *f(t-T)u(t-T).*
- If F(s) is the Laplace Transform of f(t)
- Then sifting theorem is defined as

$$\boldsymbol{L[f(t-T)u(t-T)] = e^{-sT}F(s)}$$

Proof

As the function is shifted by T, lower limit is also shifted to T

$$L[f(t-T)u(t-T)] = \int_{T}^{\infty} f(t-T)u(t-T)e^{-st}\,dt$$

$$Let\ r = t-T$$

$$t = T, r = 0$$

$$t = \infty, r = \infty$$

$$dr = dt$$

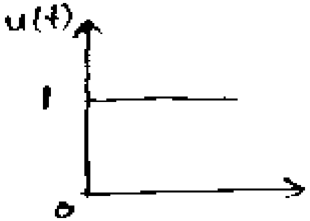
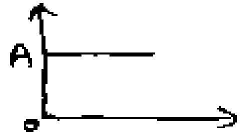
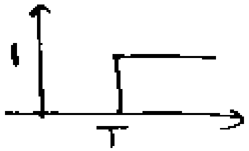
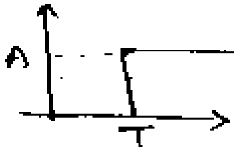
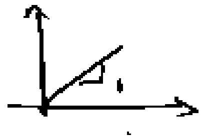
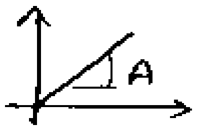
$$L[f(t-T)u(t-T)] = \int_{c=0}^{\infty} f(r)u(r)e^{-s(c+T)}\,dr$$

$$L[f(t-T)u(t-T)] = \int_{c=0}^{\infty} f(r)u(r)e^{-sc}e^{-sT}dr$$

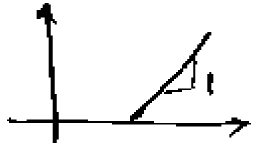
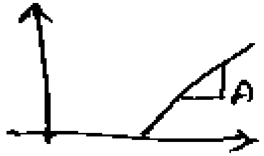
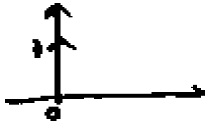
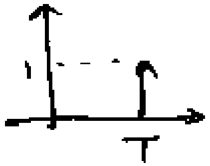
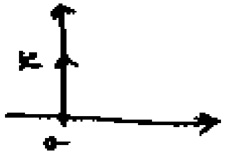
$$L[f(t-T)u(t-T)] = e^{-sT}\int_{c=0}^{\infty} f(r)u(r)e^{-sc}dr$$

$$\boldsymbol{L[f(t-T)u(t-T)] = e^{-sT}F(S)}$$

Laplace Transform of Standard Functions

	Function	F(S)	Waveform
1)	$u(t)$	$\frac{1}{S}$	
2)	$A.u(t)$	$\frac{A}{S}$	
3)	$u(t-T)$	$\frac{e^{-ST}}{S}$	
4)	$A\,u(t-T)$	$\frac{A\,e^{-ST}}{S}$	
5)	$r(t) = t\,u(t)$	$\frac{1}{S^2}$	
6)	$A.t.u(t) = A.r(t)$	$\frac{A}{S^2}$	

Laplace Transform of Standard Functions

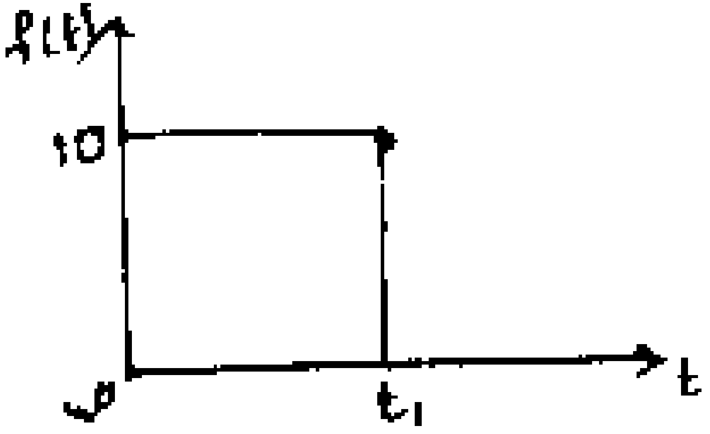
7)	$r(t-T) = (t-T)\cdot u(t-T)$	$\frac{e^{-ST}}{S^2}$	
8)	$A\cdot r(t-T) = A\cdot(t-T)\,u(t-T)$	$A\cdot\frac{e^{-ST}}{S^2}$	
9)	$\delta(t)$	1	
10)	$\delta(t-T)$	e^{-ST}	
11)	$K\,\delta(t)$	K	

Some Important Laplace Transform

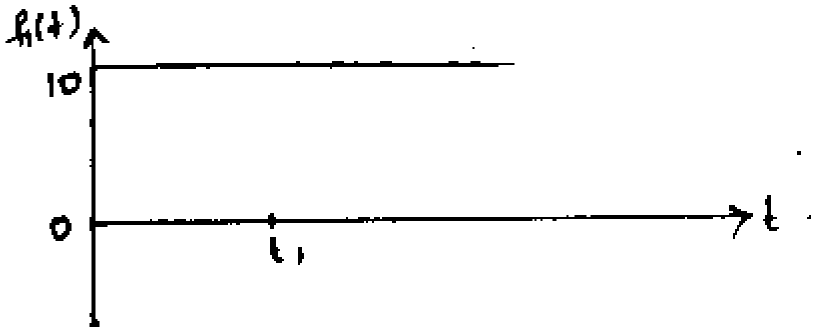
$f(t)$	$f(s)$	$f(t)$	$f(s)$
1	$\frac{1}{s}$	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
constant K	$\frac{K}{s}$	$e^{-at}\cos\omega t$	$\frac{(s+a)}{(s+a)^2+\omega^2}$
t	$\frac{1}{s^2}$	$\sinh\omega t$	$\frac{\omega}{s^2-\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$	$\cosh\omega t$	$\frac{s}{s^2-\omega^2}$
e^{-at}	$\frac{1}{(s+a)}$	te^{-at}	$\frac{1}{(s+a)^2}$
e^{at}	$\frac{1}{(s-a)}$	$1-e^{-at}$	$\frac{a}{s(s+a)}$
$e^{-at}t^n$	$\frac{n!}{(s+a)^{n+1}}$	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$		

Waveform Synthesis Using Standard Time Signals

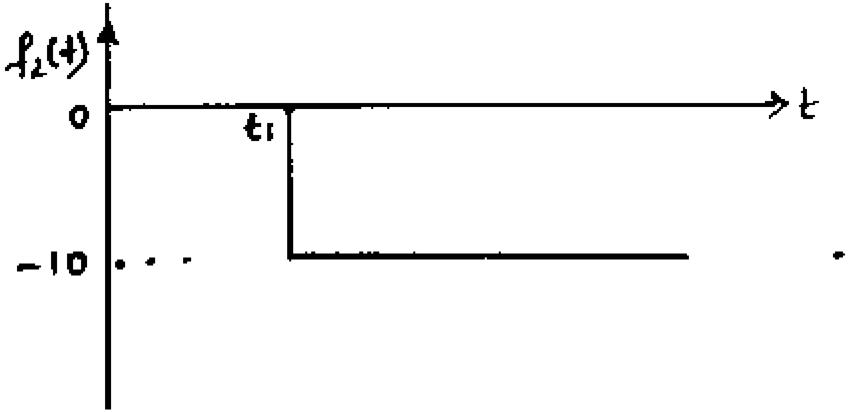
Problem1: Find the Laplace Transform of the given function.



Solution



$$f_1(t) = 10u(t)$$



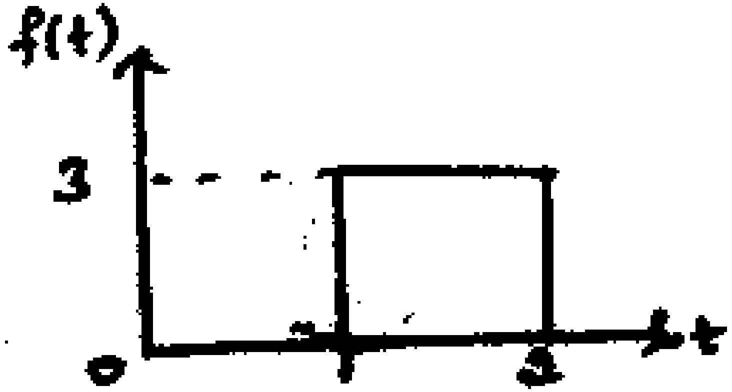
$$f_2(t) = 10u(t - t_1)$$

$$f(t) = f_1(t) + f_2(t)$$

$$f(t) = 10u(t) - 10u(t - t_1)$$

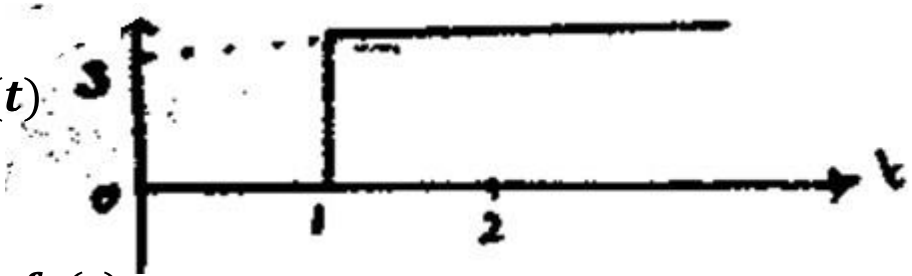
$$F(s) = \frac{10}{s} - \frac{10}{s} e^{-st_1}$$

Problem2: Find the Laplace Transform of the given function.

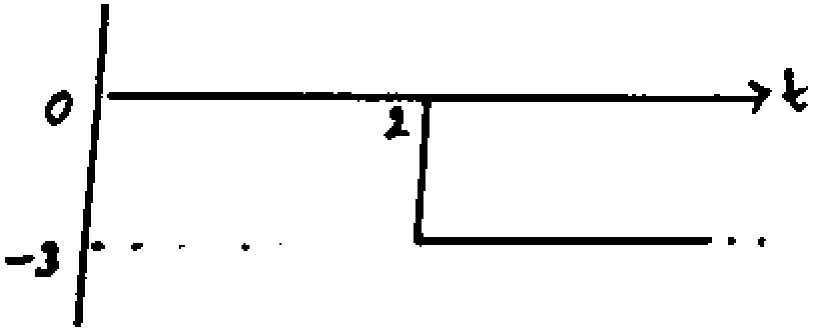


Solution

$f_1(t)$



$f_2(t)$



$$f_1(t) = 3u(t-1)$$

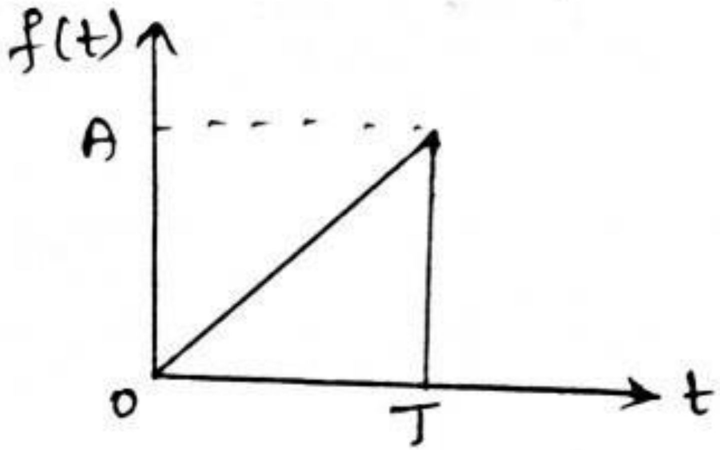
$$f_2(t) = -3u(t-2)$$

$$f(t) = f_1(t) + f_2(t)$$

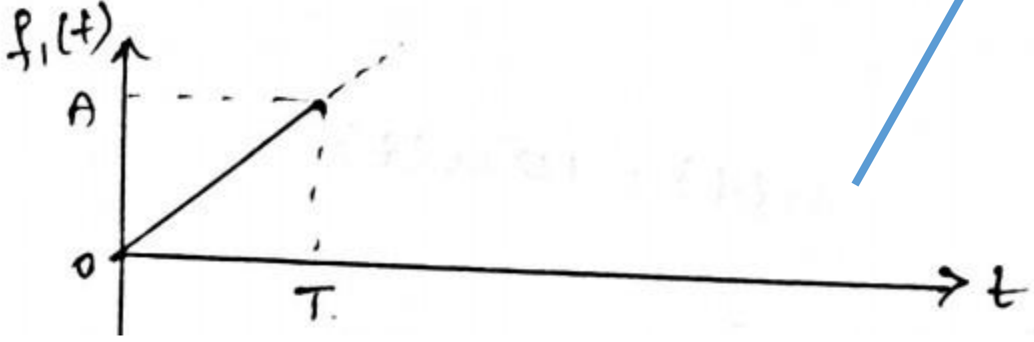
$$f(t) = 3u(t-1) - 3u(t-2)$$

$$F(s) = \frac{3}{s}e^{-s} - \frac{3}{s}e^{-2s}$$

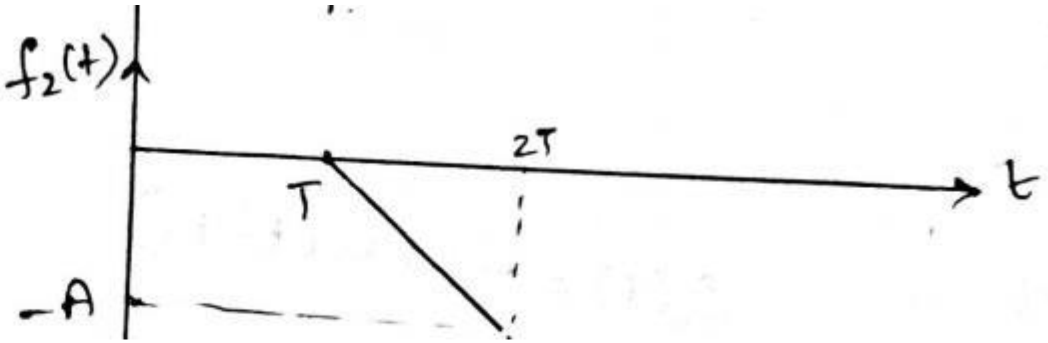
Problem3: Obtain the Laplace Transform of the given waveform



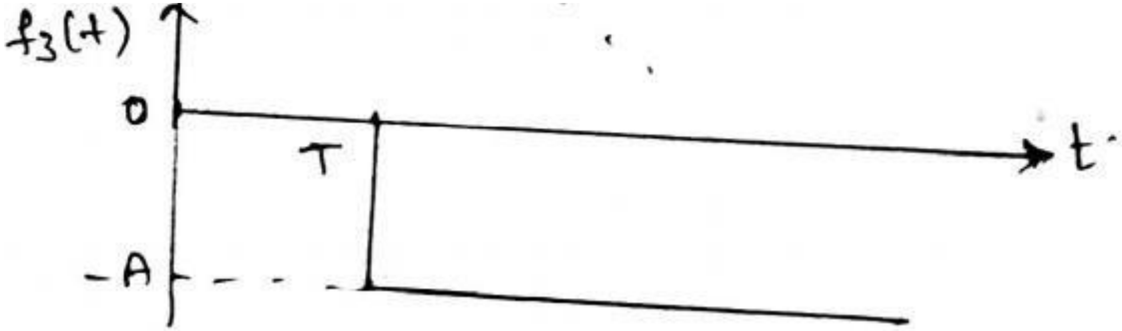
Solution



$$f_1(t) = \frac{A}{T} r(t)$$



$$f_2(t) = \frac{-A}{T} r(t - T)$$



$$f_3(t) = -Au(t - T)$$

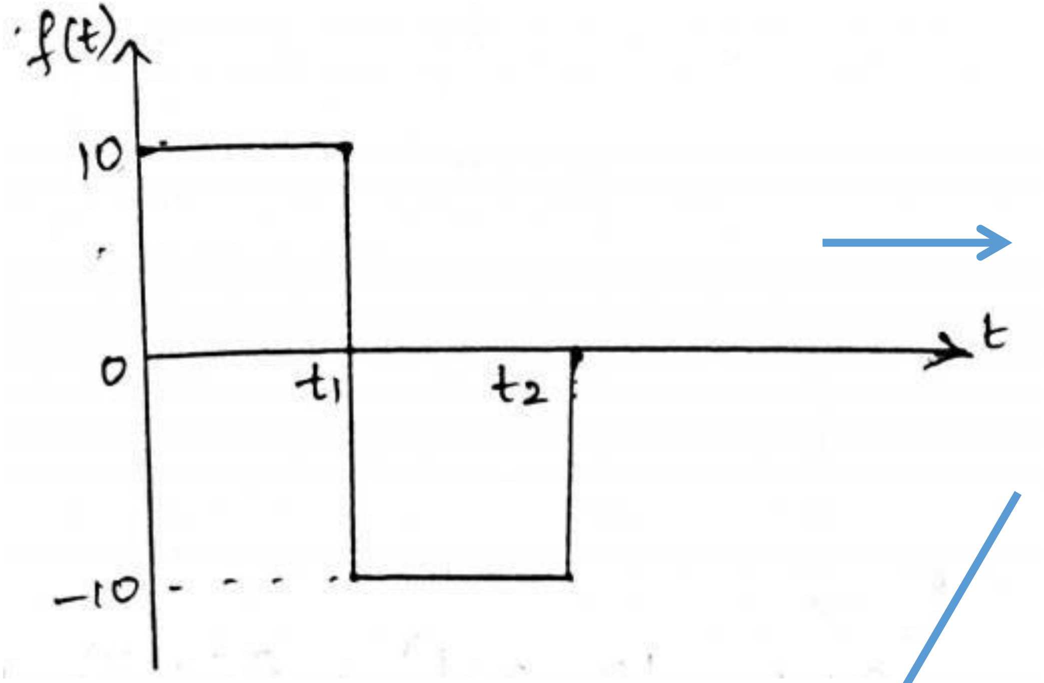
$$f(t) = f_1(t) + f_2(t) + f_3(t)$$

$$f(t) = \frac{A}{T} r(t) - \frac{A}{T} r(t-T) - Au(t-T)$$

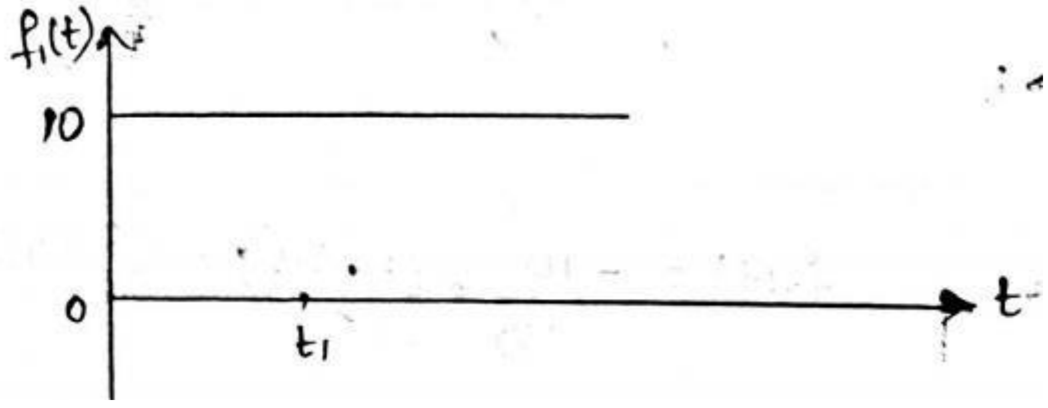
$$F(s) = \frac{A}{Ts^2} - \frac{A}{Ts^2} e^{-sT} - \frac{A}{s} e^{-sT}$$

$$F(s) = \frac{A}{Ts^2} (1 - e^{-sT} - s\, e^{-sT})$$

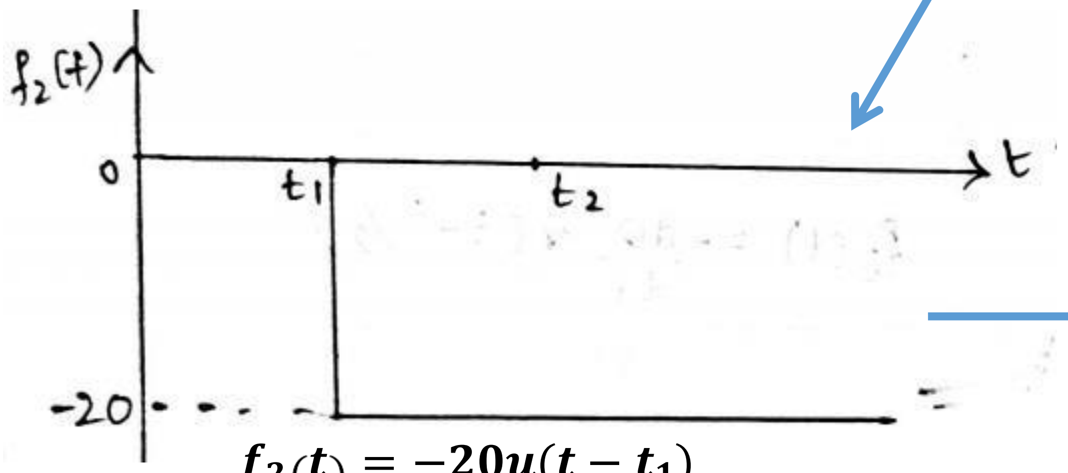
Problem4: Obtain the Laplace Transform of the given waveform



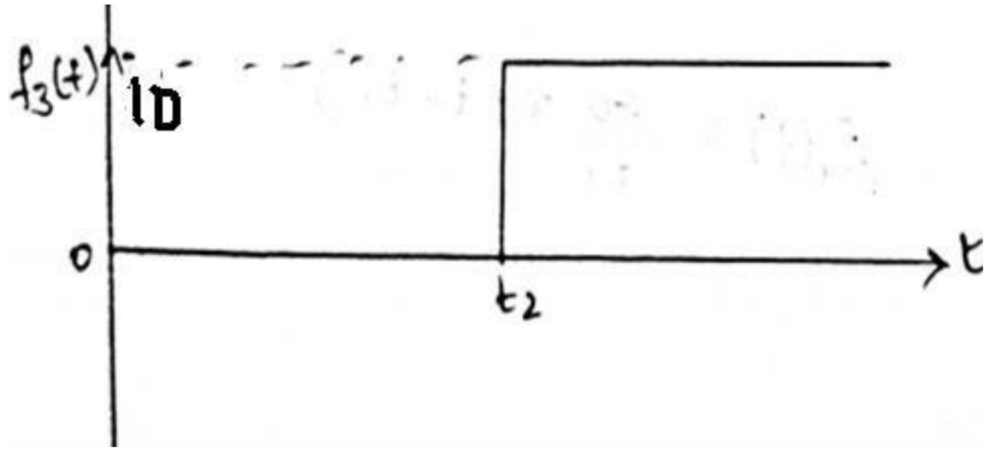
Solution



$$f_1(t) = 10u(t)$$



$$f_{2}(t) = -20u(t - t_1)$$



(t)

f_3

$$= \boldsymbol{10u(t - t_2)}$$

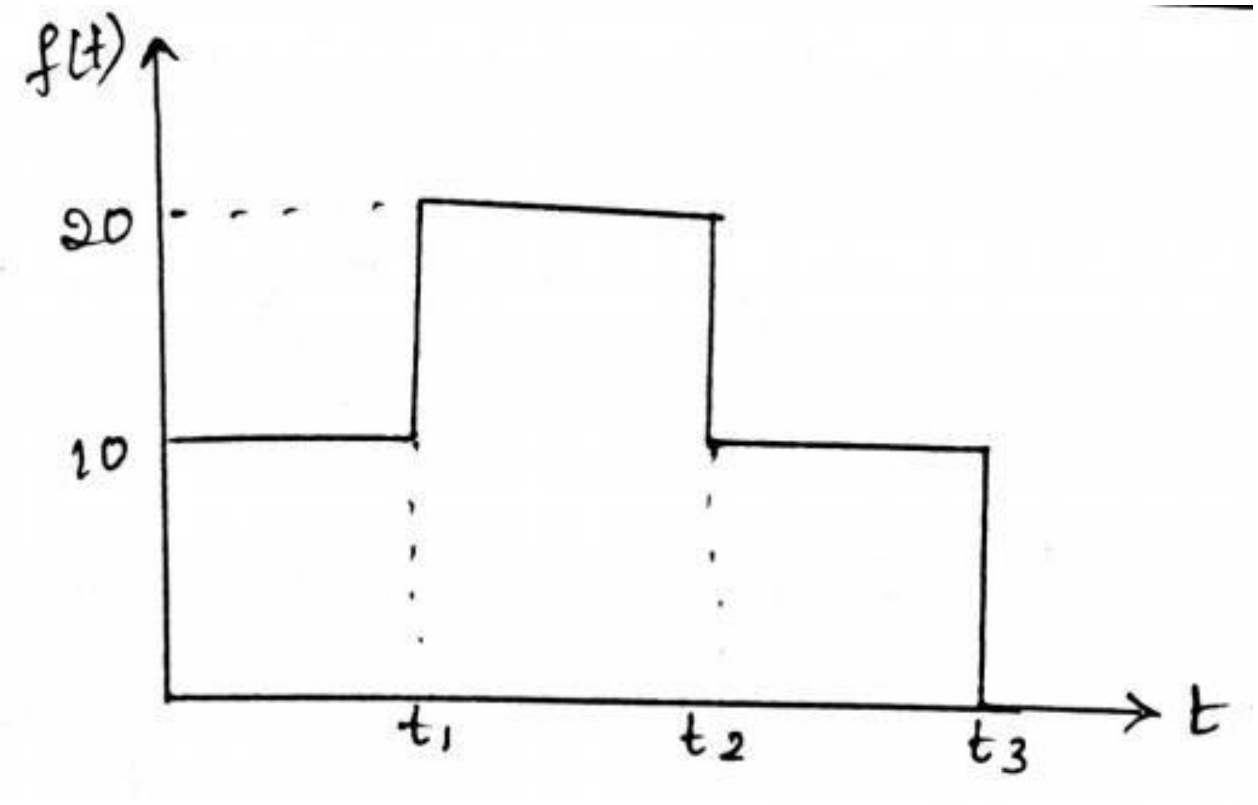
$$f(t) = f_1(t) + f_2(t) + f_3(t)$$

$$f(t) = 10u(t) - 20u(t - t_1) + 10u(t - t_2)$$

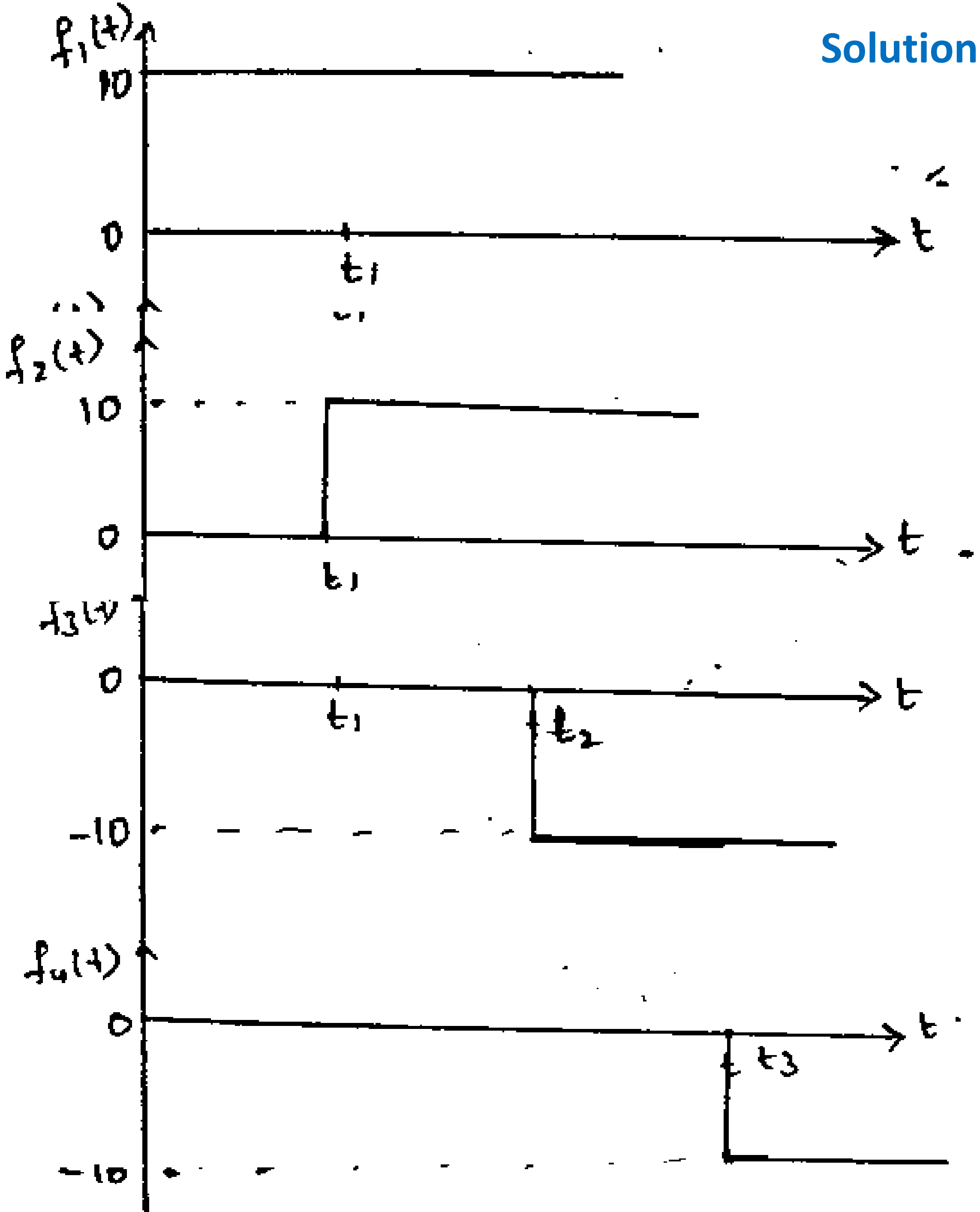
$$F(s) = \frac{10}{s} - \frac{20}{s}e^{-st_1} + \frac{10}{s}e^{-st_2}$$

$$F_{(s)} = \frac{10}{s}(1 - 2e^{-st1} + e^{-st2})$$

Problem5: **Obtain the Laplace Transform of the given waveform**



Solution



$$f_1(t) = 10\,u(t)$$

$$f_2(t) = 10\,u(t - t_1)$$

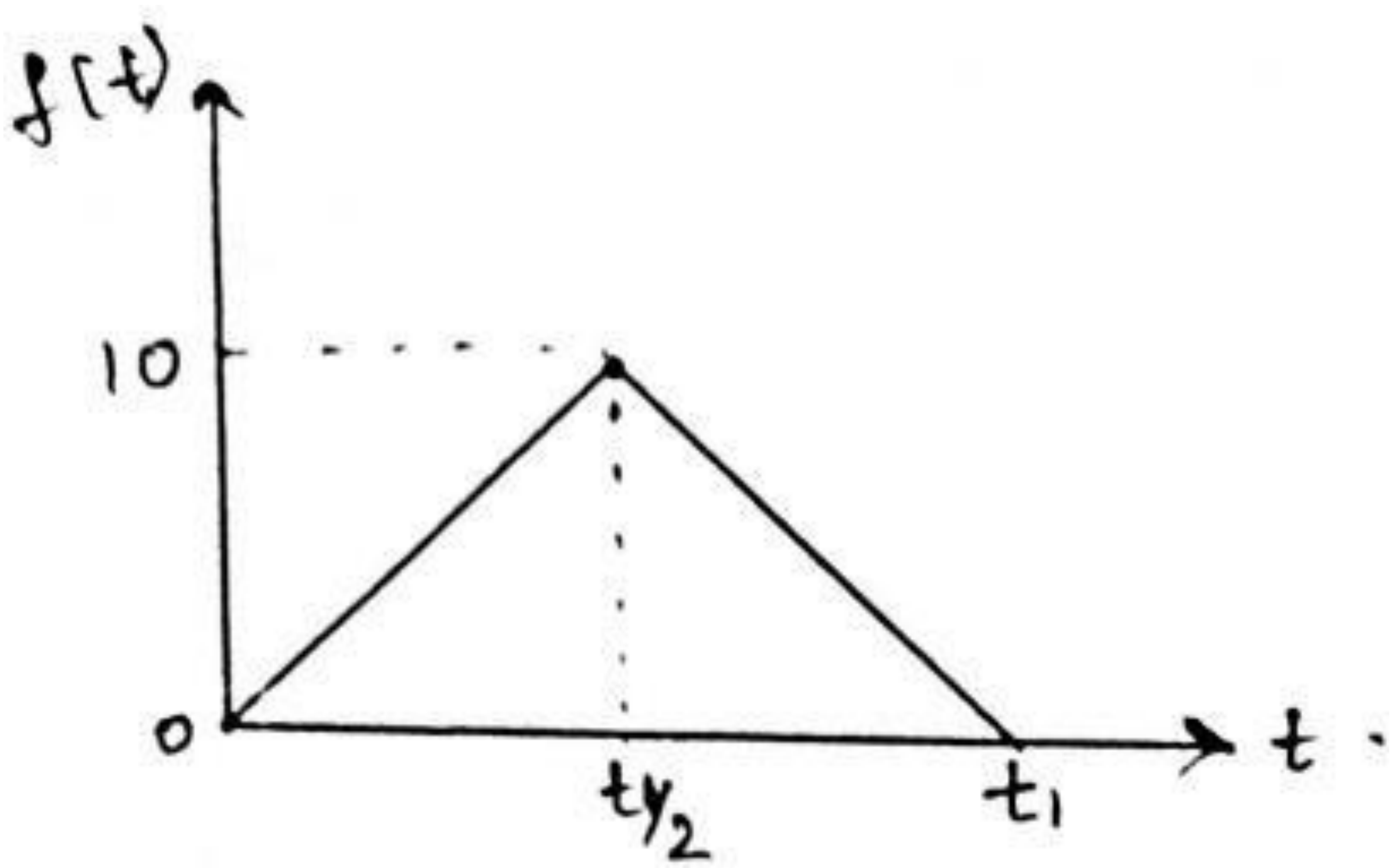
$$f_3(t) = -10\,u(t - t_2)$$

$$f_4(t) = -10\,u(t - t_3)$$

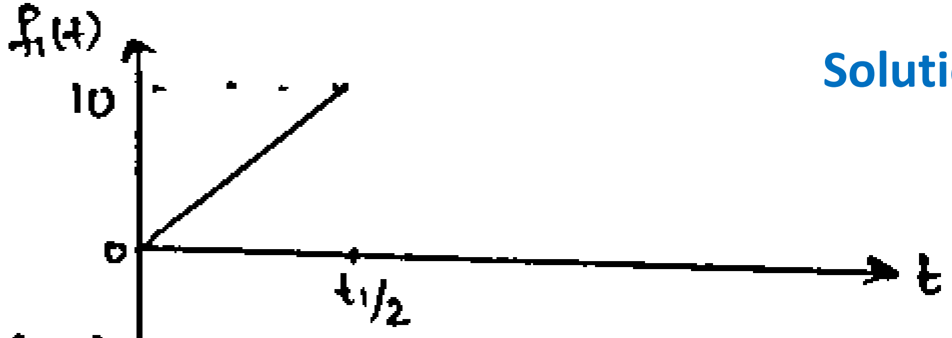
$$f(t) = 10u(t) + 10u(t-t_1) - 10u(t-t_2) - 10u(t-t_3)$$

$$F(s) = \frac{10}{s} + \frac{10}{s}e^{-st_1} - \frac{10}{s}e^{-st_2} - \frac{10}{s}e^{-st_3}$$

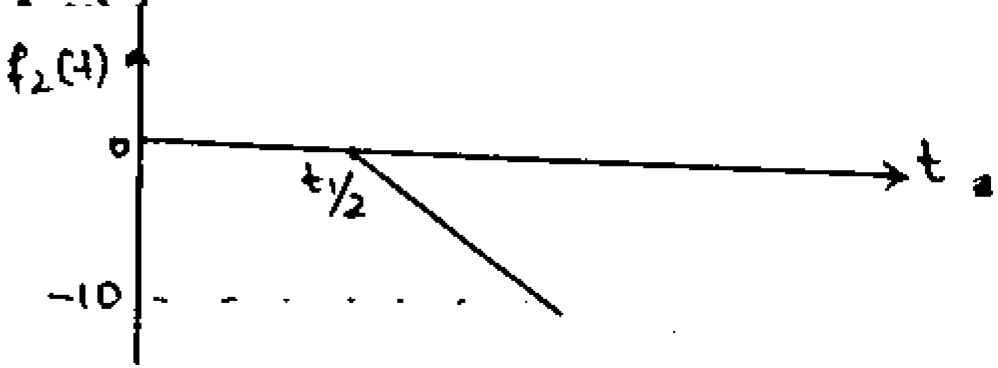
Problem6: **Obtain the Laplace Transform of the given waveform**



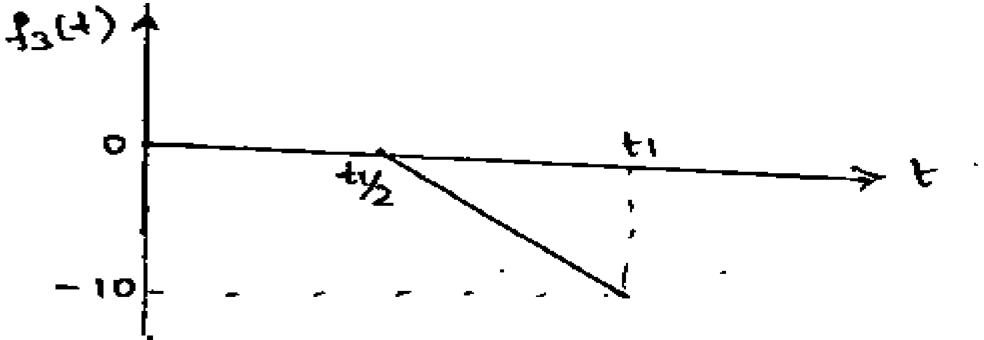
Solution



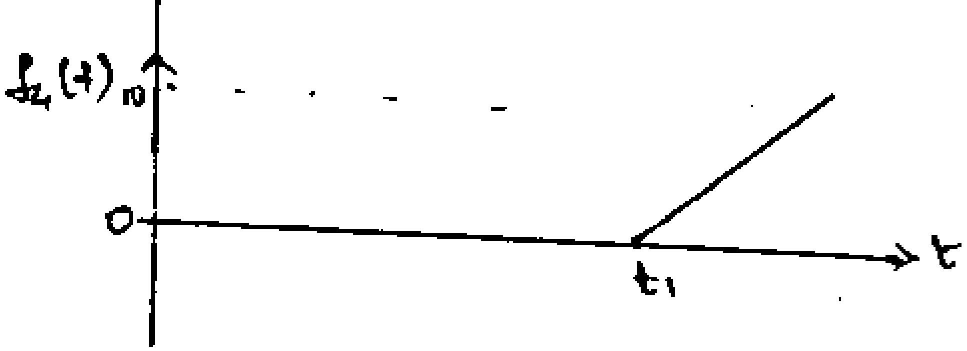
$$f_1(t) = \frac{10}{t_1/2}\, r(t) = \frac{20}{t_1}\, r(t).$$



$$f_2(t) = -\frac{10}{t_1/2}\, r\left(t - t_1/2\right) = \frac{-20}{t_1}\, r\left(t - t_1/2\right)$$



$$f_3(t) = -\frac{20}{t_1}\, r\left(t - t_1/2\right)$$



$$f_4(t) = \frac{20}{t_1}\, r\left(t - t_1\right)$$

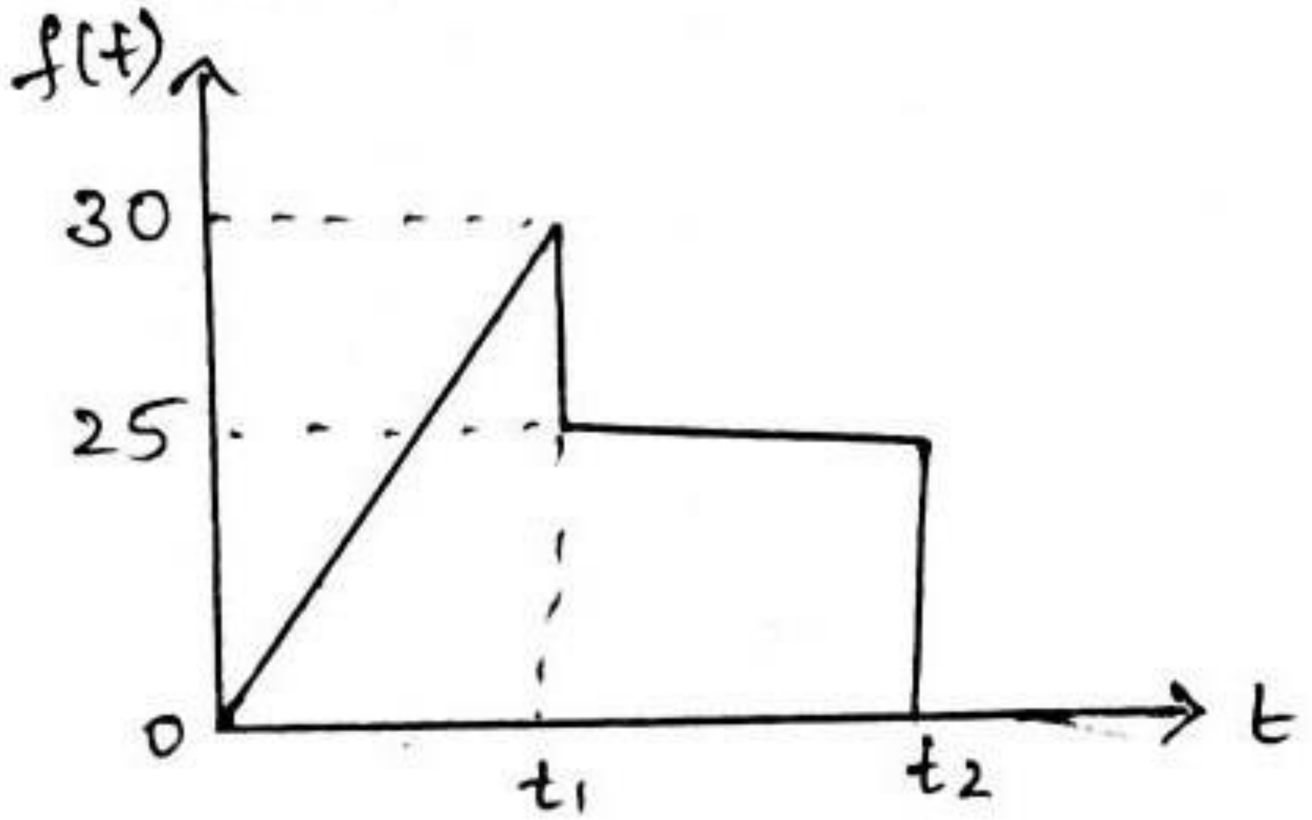
$$f(t) = f_1(t) + f_2(t) + f_3(t) + f_4(t)$$

$$= \frac{20}{t_1} r(t) - \frac{20}{t_1} r(t - t_1/2) - \frac{20}{t_1} r(t - t_1/2) + \frac{20}{t_1} r(t - t_1)$$

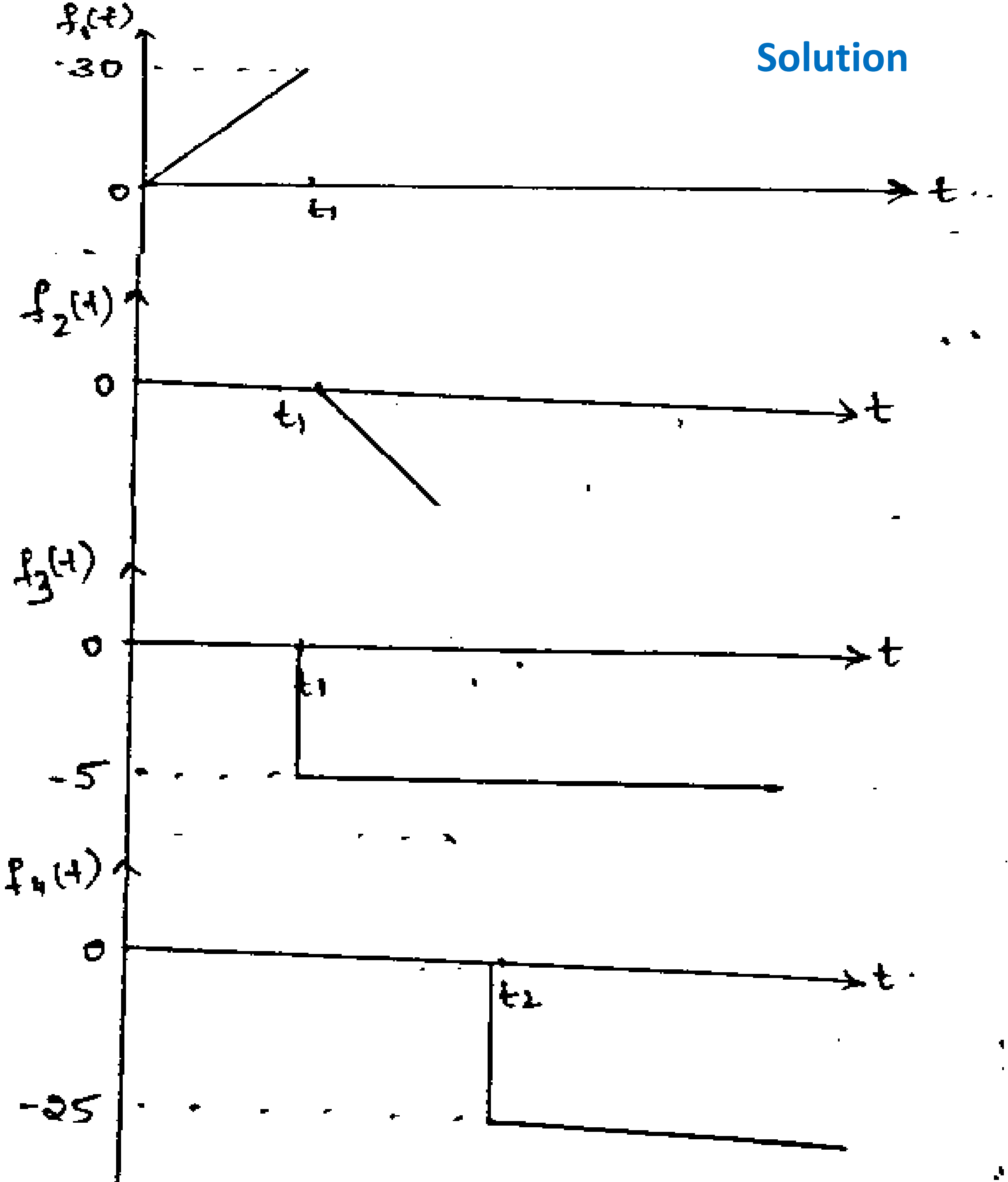
$$= \frac{20}{t_1} r(t) - \frac{40}{t_1} r(t - t_1/2) + \frac{20}{t_1} r(t - t_1)$$

$$\left\langle F(s) = \frac{20}{t_1 s^2} - \frac{40}{t_1 s^2} \cdot e^{-st_1/2} + \frac{20}{t_1 s^2} \cdot e^{-st_1} \right\rangle$$

Problem7: **Obtain the Laplace Transform of the given waveform**



Solution



$$f_1(t) = \frac{30}{t_1} r(t)$$

$$f_2(t) = -\frac{30}{t_1} r(t-t_1)$$

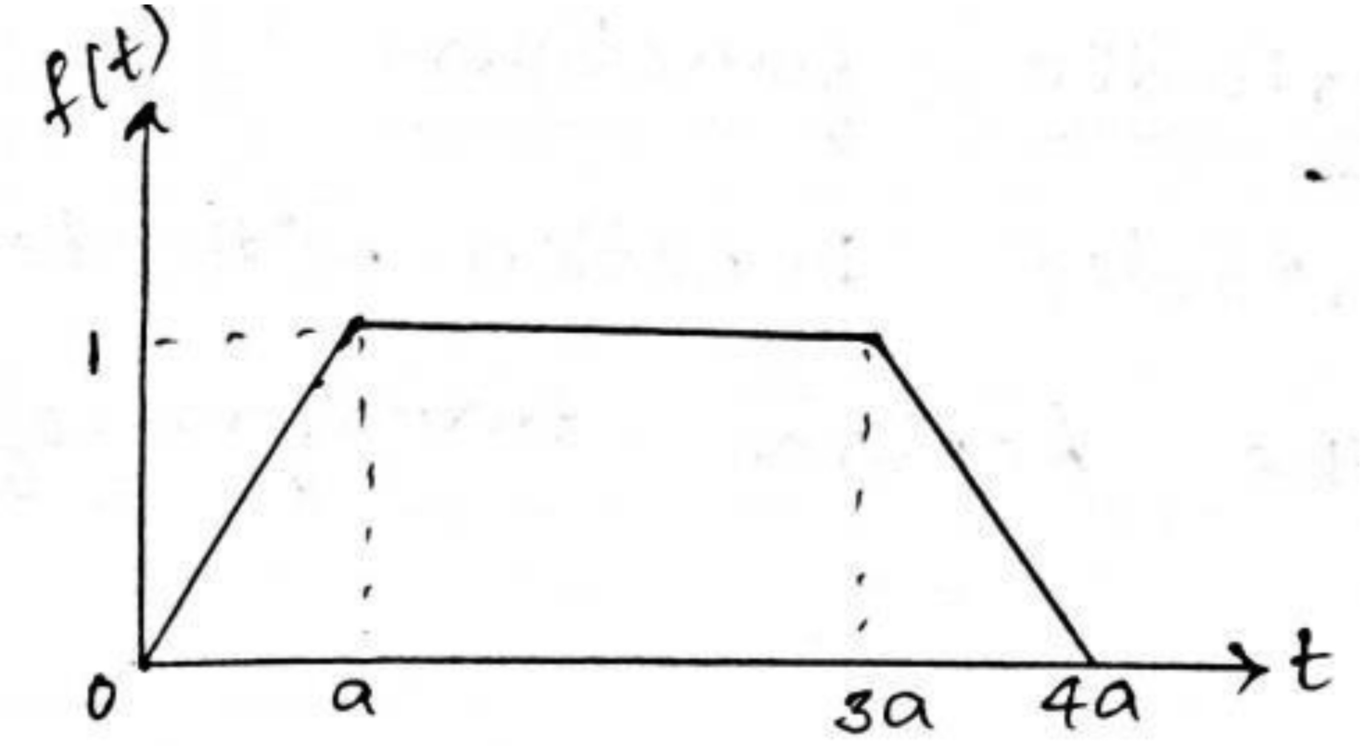
$$f_3(t) = -5u(t-t_1)$$

$$f_4(t) = -25u(t-t_2)$$

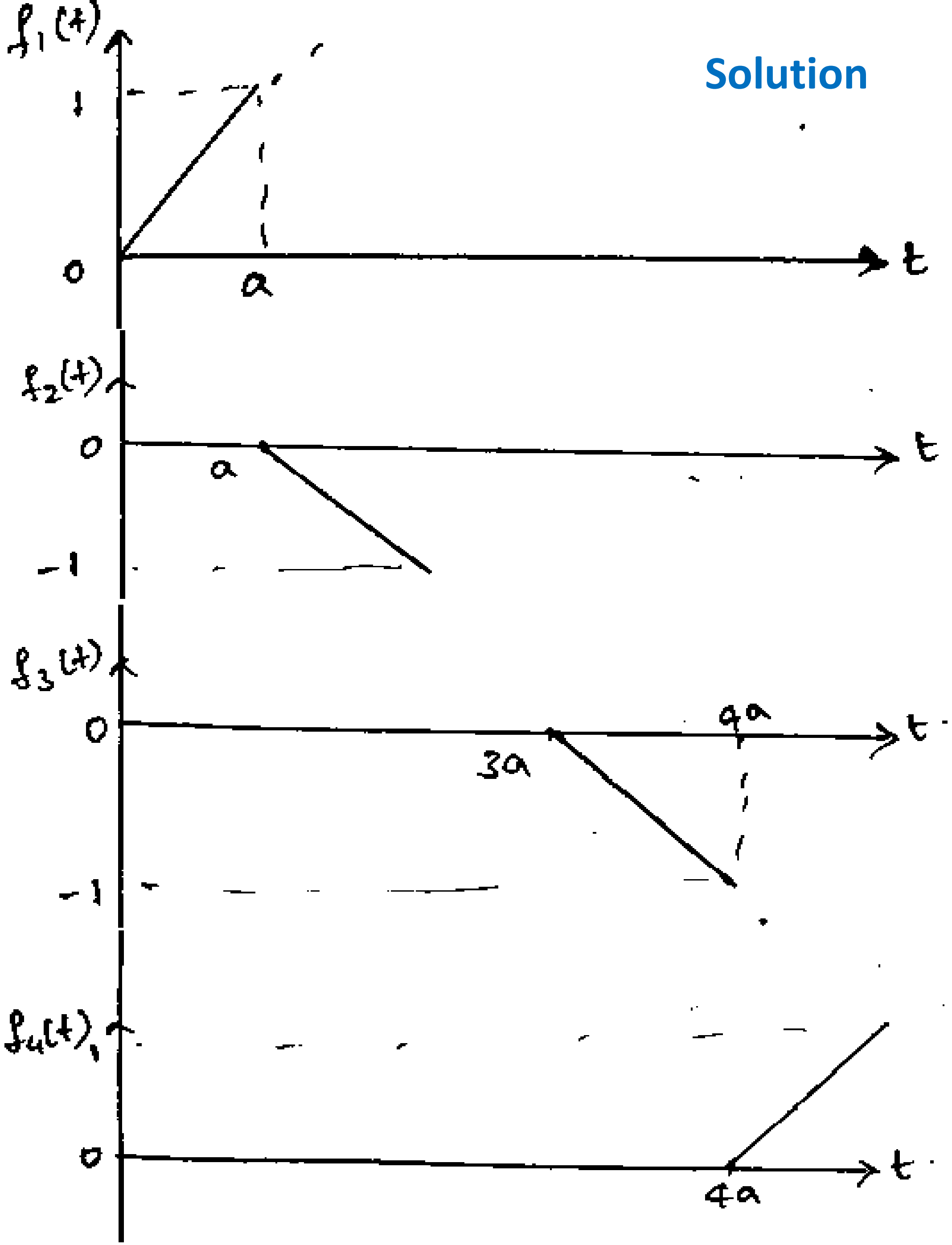
$$f(t) = \frac{30}{t_1} r(t) - 5u(t-t_1) - \frac{30}{t_1} r(t-t_1) - 25u(t-t_2)$$

$$\left\langle F(S) = \frac{30}{t_1 S^2} - \frac{5}{S} e^{-St_1} - \frac{30}{t_1 S^2} e^{-St_1} - \frac{25}{S} e^{-St_2} \right\rangle$$

Problem8: **Obtain the Laplace Transform of the given waveform**



Solution



$$f_1(t) = \frac{1}{a} r(t)$$

$$f_2(t) = -\frac{1}{a} r(t-a)$$

$$f_3(t) = -\frac{1}{a} r(t-3a)$$

$$f_4(t) = \frac{1}{a} r(t-4a)$$

$$f(t) = \frac{1}{a} r(t) - \frac{1}{a} r(t-a) - \frac{1}{a} r(t-3a) + \frac{1}{a} r(t-4a)$$

$$\left\langle F(s) = \frac{1}{as^2} - \frac{1}{as^2} e^{-sa} - \frac{1}{as^2} e^{-s3a} + \frac{1}{as^2} e^{-s4a} . \right\rangle$$

Laplace Transform Of Periodic Function

- Laplace transform of the periodic function with period T is $\frac{1}{1-e^{-sT}}$ times the Laplace transform of the first cycle.

Proof

Let f(t) is the periodic function of time period 'T'

Let $f_1(t), f_2(t), f_3(t)$ be the function describing the first , second, third cycle etc.

$$f(t) = f_1(t) + f_2(t) + f_3\ (t) + \ \dots\dots$$

$$f(t) = f_1\ (t) + f_1\ (t - T\ ) + f_1(t\ - 2T) + \dots\dots$$

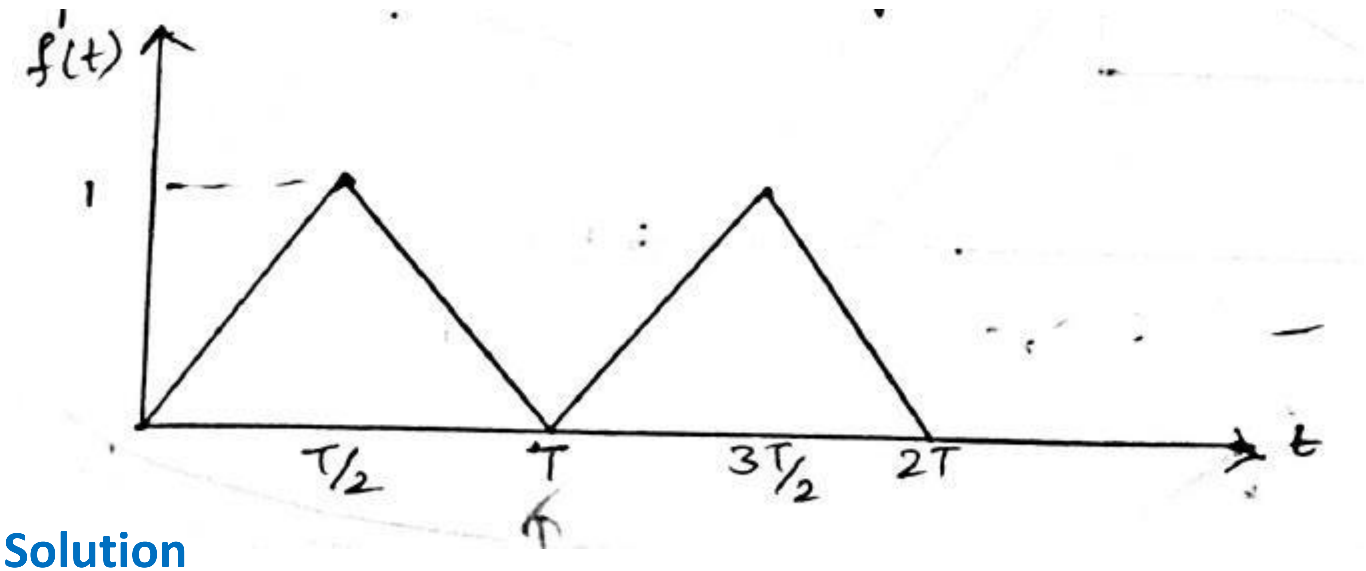
$$F(s) = F_1\ (S) + F_1\ (S)e^{-sT} + F_1\ (S)e^{-2sT} + \cdots$$

$$F(s) = F_1(S)(1 + e^{-sT} + e^{-2sT} + \dots..)$$

According To Geometric Series $\quad (1 + e^{-sT} + e^{-2sT} + \ \dots..) = \dfrac{1}{1 - e^{-sT}}$

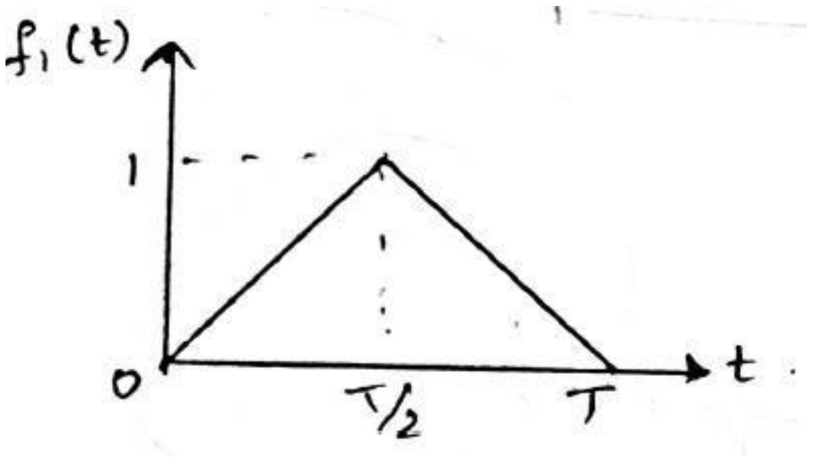
$$F(s) = F_1\ (S)\frac{1}{1 - e^{-sT}}$$

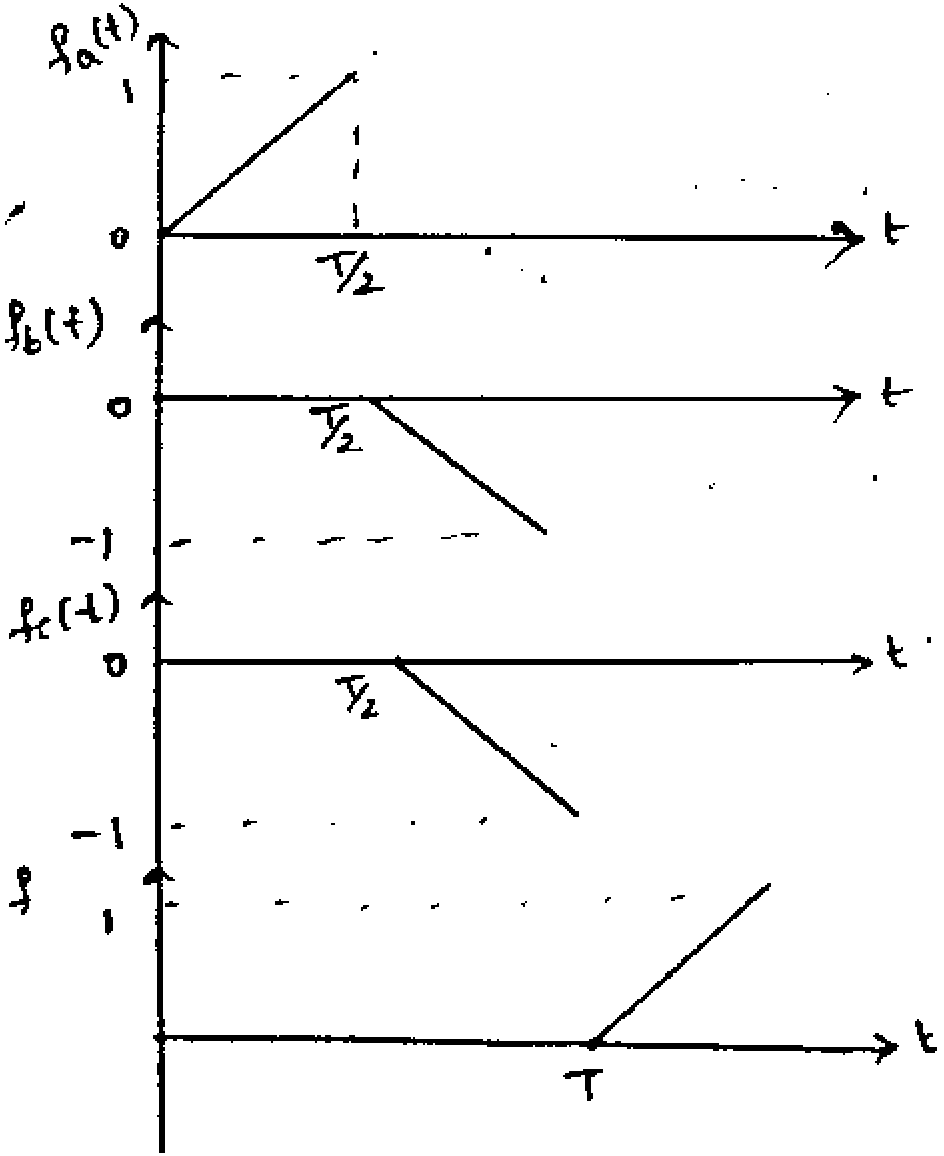
Problem1: **Synthesis the waveform shown and find the Laplace Transform of the periodic waveform.**



Solution

Consider the first cycle from 0 to T





$$f_a(t) = \frac{2}{T} r(t)$$

$$f_b(t) = -\frac{2}{T} r(t - T/2)$$

$$f_c(t) = -\frac{1}{T/2} r(t - T/2)$$

$$f_c(t) = -\frac{2}{T} r(t - T/2)$$

$$f_d(t) = \frac{2}{T} r(t - T)$$

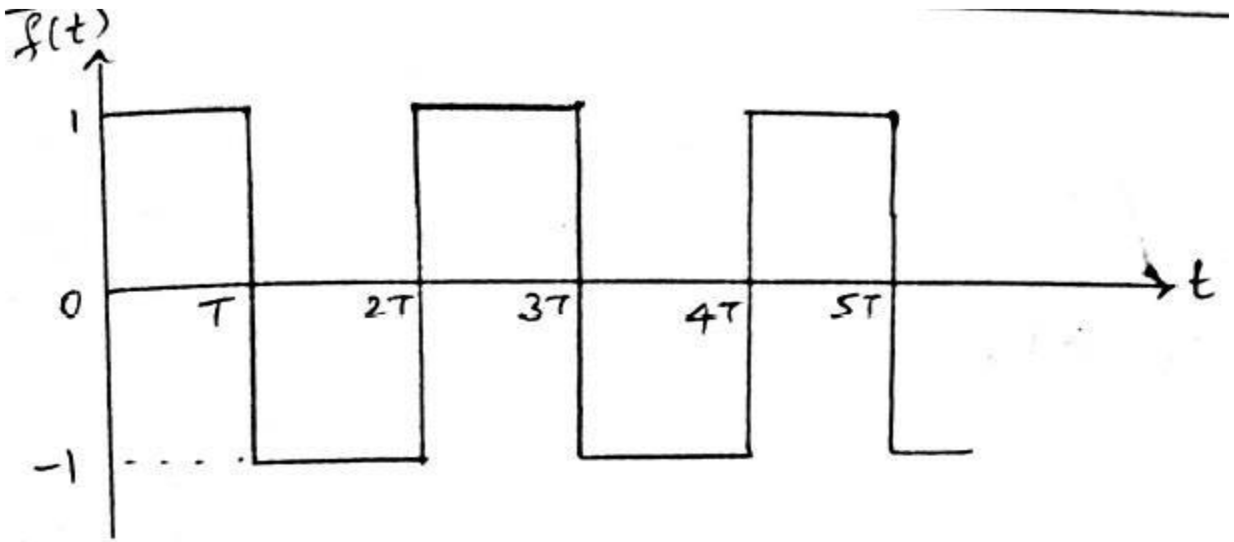
$$f_1(t) = f_a(t) + f_b(t) + f_c(t) + f_d(t)$$

$$= \frac{2}{T} r(t) - \frac{2}{T} r(t - T/2) - \frac{2}{T} r(t - T/2) + \frac{2}{T} r(t - T)$$

$$F_1(s) = \frac{2}{s^2 T} - \frac{4}{s^2 T} e^{-sT/2} + \frac{2}{s^2 T} e^{-sT}$$

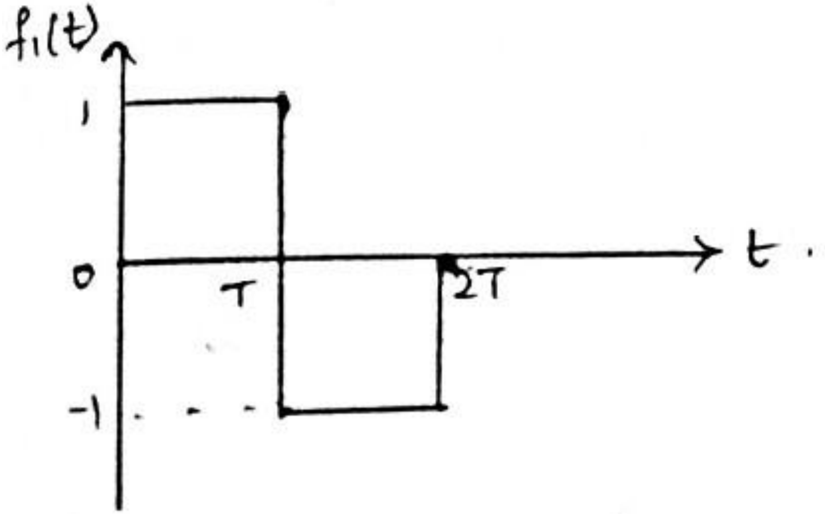
$$F(s) = \frac{F_1(s)}{1 - e^{-Ts}} = \frac{\frac{2}{Ts^2}\left[1 - 2e^{-sT/2} + e^{-sT}\right]}{1 - e^{-sT}} \quad //$$

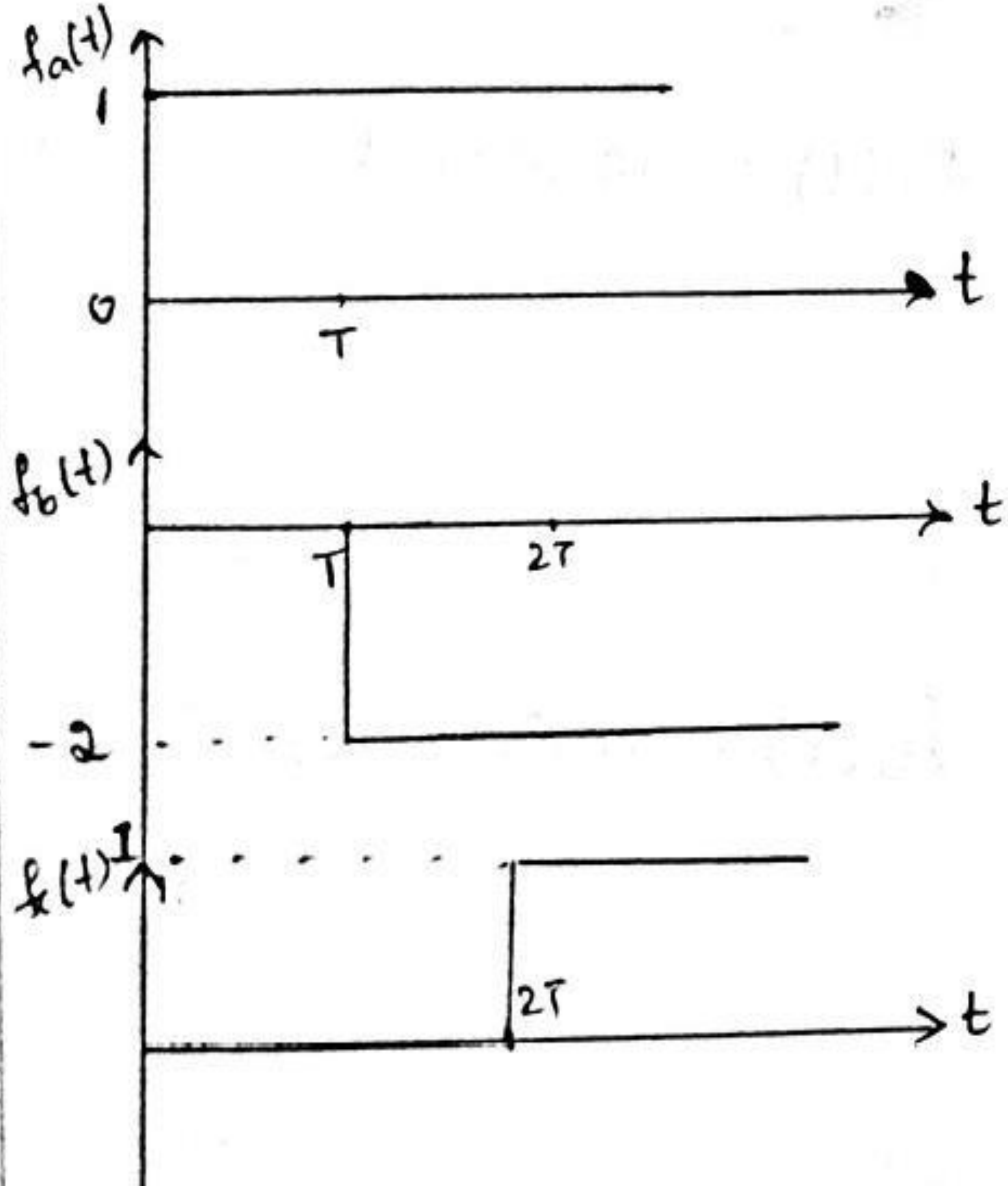
Problem2: **Synthesis the waveform shown and find the Laplace Transform of the periodic waveform.**



Solution

Consider the first cycle from 0 to 2T





$$f_a(t) = u(t)$$

$$f_b(t) = -2u(t-T)$$

$$f_c(t) = u(t-2T)$$

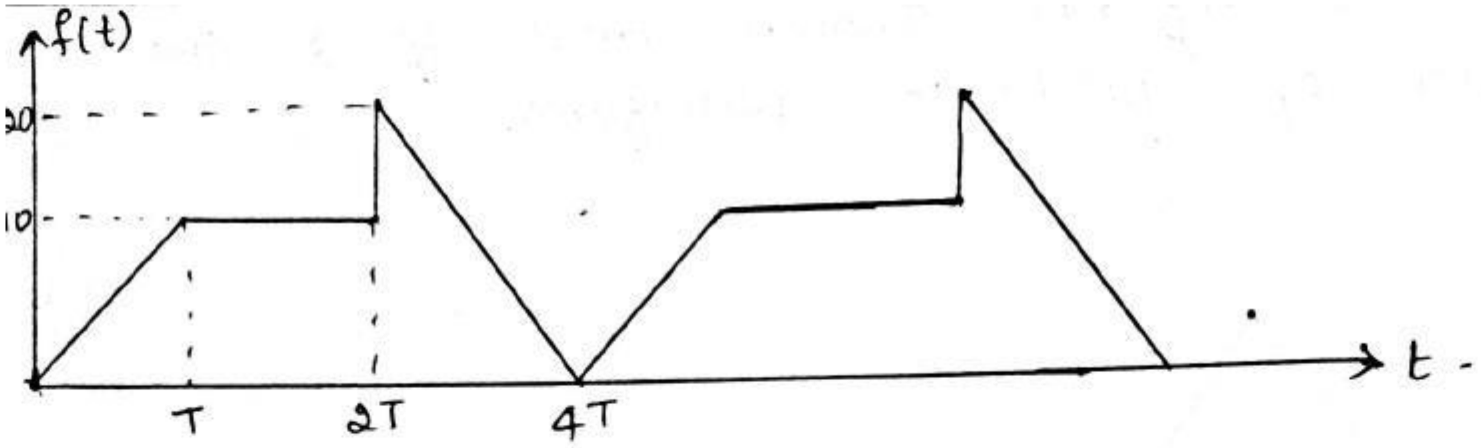
$$f_1(t) = u(t) - 2u(t-T) + u(t-2T)$$

$$= \frac{1}{s} - \frac{2}{s} e^{-sT} + \frac{1}{s} e^{-s2T}.$$

$$F_1(s) = \frac{1}{s}\left[1 - e^{-sT} + e^{-2sT}\right]$$

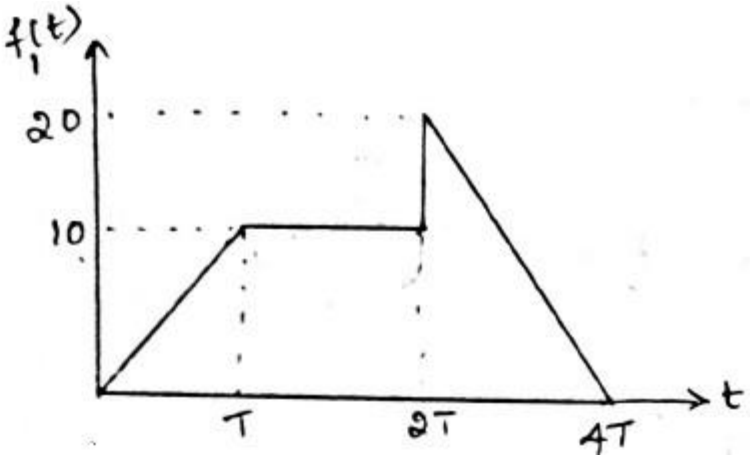
$$\left\langle F(s) = \frac{\frac{1}{s}\left[1 - e^{-sT} + e^{-2sT}\right]}{1 - e^{-2sT}} \right\rangle$$

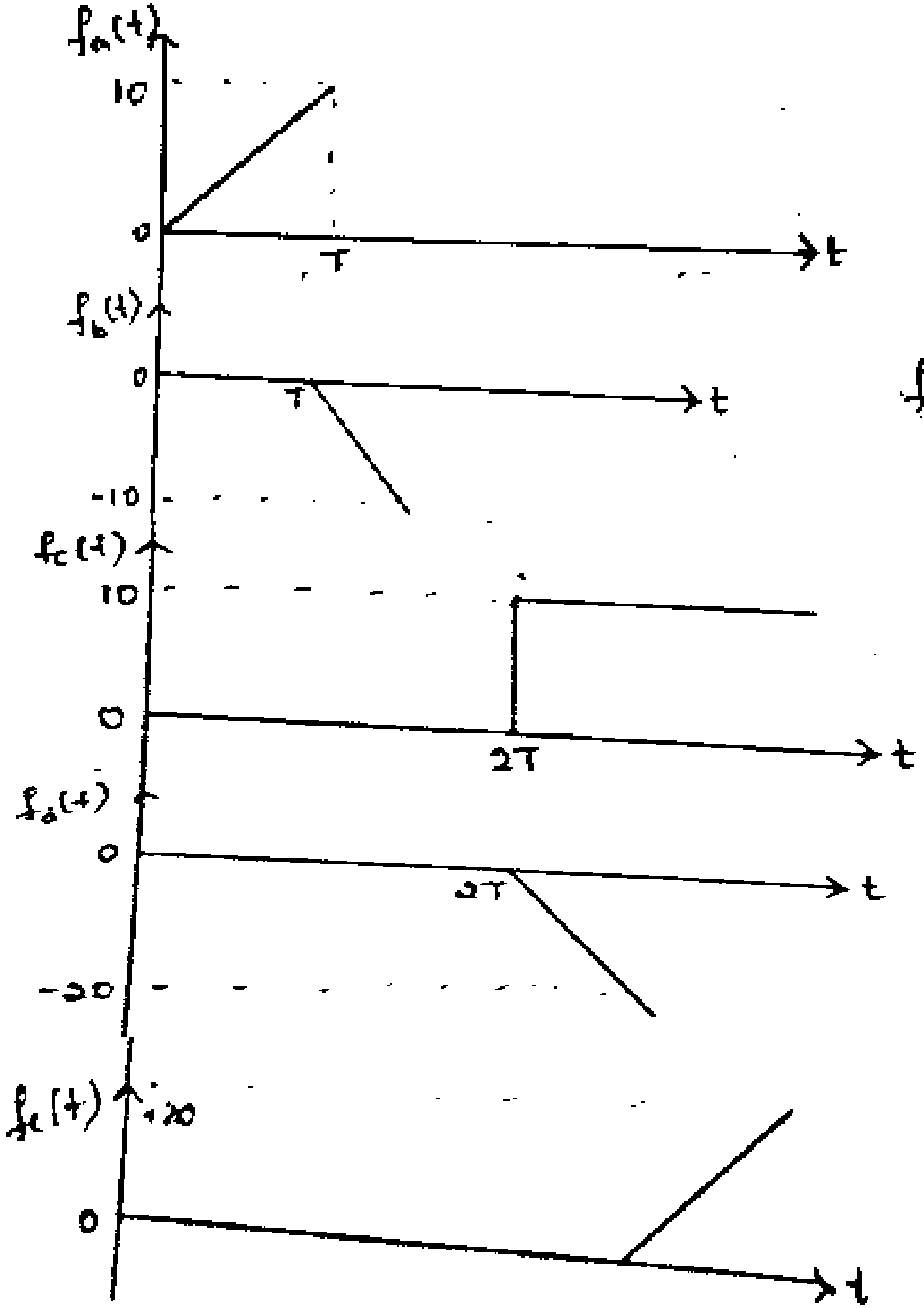
Problem3: Synthesis the waveform shown and find the Laplace Transform of the periodic waveform.



Solution

Consider the first cycle from 0 to 4T





$$f_a(t) = \frac{10}{T} r(t)$$

$$f_b(t) = \frac{-10}{T} r(t-T)$$

$$f_c(t) = 10\, u(t-2T)$$

$$f_d(t) = \frac{-20}{2T} r(t-2T)$$

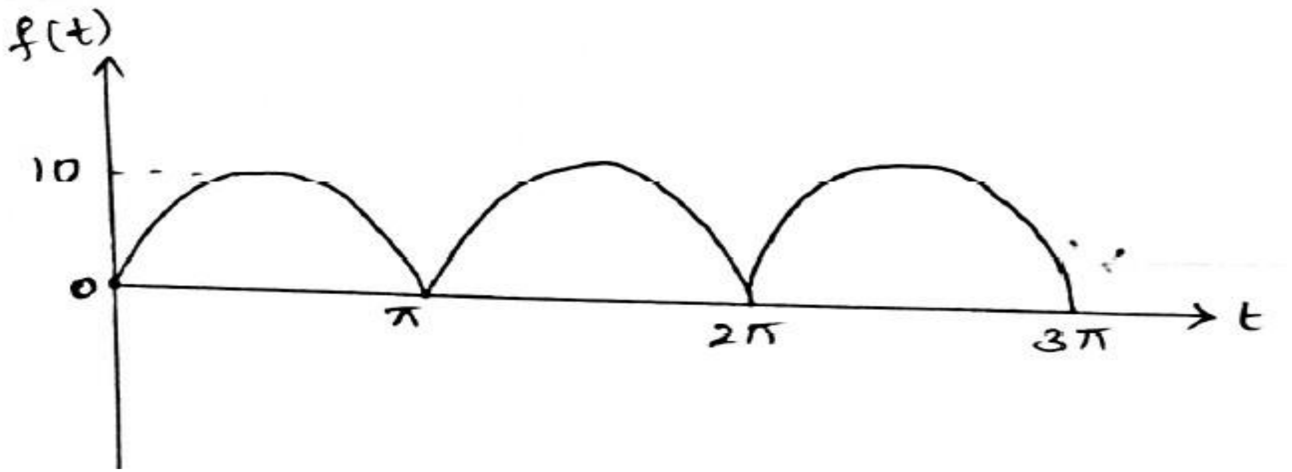
$$f_e(t) = \frac{20}{2T} r(t-4T)$$

$$f_1(t) = \frac{10}{T} r(t) - \frac{10}{T} r(t-T) + 10u(t-2T) - \frac{20}{2T} r(t-2T) + \frac{20}{2T} r(t-4T)$$

$$F_1(s) = \frac{10}{Ts^2} - \frac{10}{Ts^2} e^{-sT} + \frac{10}{s} e^{-2sT} - \frac{20}{2Ts^2} e^{-2sT} + \frac{20}{2Ts^2} e^{-4sT}.$$

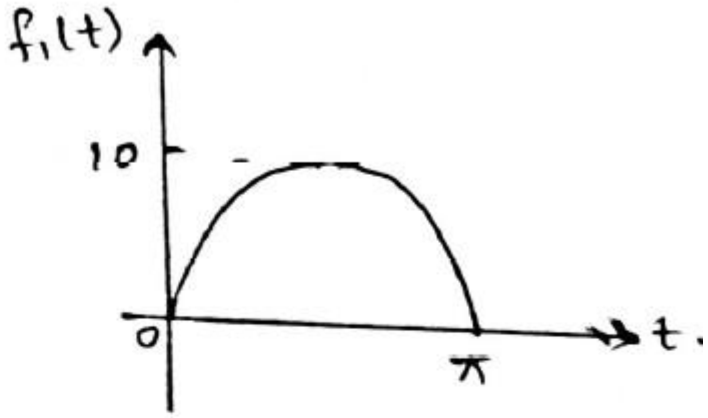
$$\left\{ F(s) = \frac{F_1(s)}{1-e^{-4sT}} \right\}$$

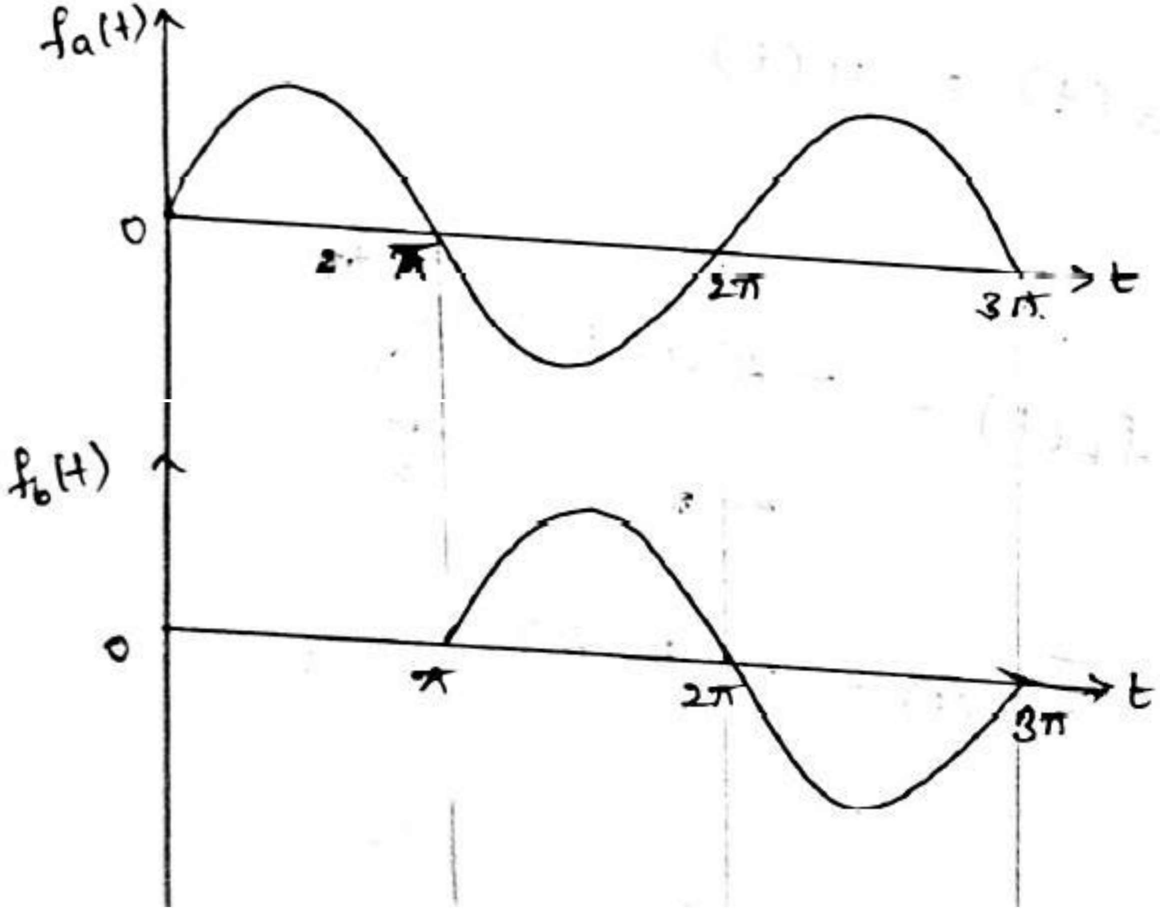
Problem4: Synthesis the waveform shown and find the Laplace Transform of the periodic waveform.



Solution

Consider the first cycle from 0 to π





$$f_a(t) = 10 \sin \omega t$$

$$f_b(t) = 10 \sin \omega (t - \pi)$$

$$f_1(t) = 10 \sin \omega t + 10 \sin \omega (t - \pi)$$

$$F_1(s) = 10 \times \frac{\omega}{s^2+\omega^2} + \frac{10 \times \omega}{s^2+\omega^2} e^{-s\pi}$$

$$F(s) = \frac{F_1(s)}{1-e^{-s\pi}}$$

$$\left\{ F(s) = \frac{10\omega}{s^2+\omega^2}\left[\frac{1+e^{-s\pi}}{1-e^{-s\pi}}\right] \right\}$$

Initial Value Theorem

- It states that the initial value of time function f(t) is obtained from its Laplace transform as given below.

$$\boldsymbol{f(o^+) = \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)}$$

- The only restriction is f(t) must be continuous.

Proof

Consider the Laplace transform of the differentiation function

$$L\left[\frac{d}{dt}\big(f(t)\big)\right] = sF(s) - f(0^-)$$

Taking limit as $s \to \infty$ on both sides

$$\lim_{s\to\infty} L\left[\frac{d}{dt}\big(f(t)\big)\right] = \lim_{s\to\infty}[sF(s) - f(0^-)] \ \ldots e(1)$$

Consider L.H.S of equation 1.

$$\lim_{s\to\infty} L\left[\frac{d}{dt}(f(t))\right] = \lim_{s\to\infty} \int_0^{\infty} e^{-st}\frac{d}{dt}(f(t))\, dt$$

$$\lim_{s\to\infty} L\left[\frac{d}{dt}(f(t))\right] = 0$$

$\therefore$ R.H.S $= 0$ in eq(1)

$$0 = \lim_{s\to\infty}[sF(s) - f(0^-)]$$

$$0 = \lim_{s\to\infty}[sF(s)] - (0^-)$$

$$(0^-) = \lim_{s\to\infty}[sF(s)]$$

But as the function f(t) is continuous

$f(0^-) = (0^+)$ i.e., the initial value of f(t)

$$\boxed{\boldsymbol{f(o^+) = \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)}}$$

Problem1: Find the initial value of $F(s) = \frac{4s+5}{(s+1)(s+3)}$

Solution

$$sF(s) = \frac{(4s+5)}{(s+1)(s+3)}$$

$$f(o^+) = \lim_{s\to\infty} sF(s)$$

$$f(o^+) = \lim_{s\to\infty} \frac{(4s+5)}{(s+1)(s+3)}$$

$$f(o^+) = \lim_{s\to\infty} \frac{s*(4+5/s)}{s\left(1+\frac{1}{s}\right)\left(1+\frac{3}{s}\right)s}$$

$$f(o^+) = 4$$

Problem2: Find the initial value of $F(s) = \frac{s+1}{(s+1)^2+3^2}$

Solution

$$sF(s) = \frac{(s+1)}{(s+1)^2+3^2}$$

$$f(o^+) = \lim_{s\to\infty} sF(s)$$

$$f(o^+) = \lim_{s\to\infty} \frac{(s+1)}{(s+1)^2+3^2}$$

$$f(o^+) = \lim_{s\to\infty} \frac{s*(s+1)}{(s^2+2s+1)+3^2}$$

$$f(o^+) = \lim_{s\to\infty} \frac{s*(1+1/s)}{s^2\left[\left(1+\frac{2}{s}+\frac{1}{s^2}\right)+\frac{3^2}{s^2}\right]}$$

$$f(o^+) = 1$$

Problem3: Find the initial value of $F(s) = \frac{A(a+s)\sin\theta + Q\cos\theta}{(s+a)^2+Q^2}$

Solution

$$sF(s) = \frac{s[A(\alpha + s)sin\ \theta + \beta cos\ \theta]}{(s+\alpha)^2 + \beta^2}$$

$$f(o^+) = \lim_{s\to\infty} sF(s)$$

$$f(o^+) = \lim_{s\to\infty} \frac{sA(\alpha + s)sin\ \theta + \ \beta cos\ \theta}{(s+\alpha)^2 + \beta^2}$$

$$f(o^+) = \lim_{s\to\infty} \frac{sA\ *s\left[\left(\frac{\alpha}{s} + 1\right) sin\ \theta + \beta cos\ \theta/s\right]}{(s^2 + 2\alpha s + \alpha^2) + \beta^2}$$

$$f(o^+) = \lim_{s\to\infty} \frac{sA\ *s\left[\left(\frac{\alpha}{s} + 1\right) sin\ \theta + \beta cos\ \theta/s\right]}{s^2\left[\left(1 + \frac{2\alpha}{s} + \frac{\alpha^2}{s^2}\right) + \frac{\beta^2}{s^2}\right]}$$

$$f(o^+) = \mathrm{Asin}\theta$$

Problem4: Find the initial value of $F(s) = \frac{10}{s+2} - \frac{4}{s+3}$

Solution

$$sF(s) = s\left[\frac{10}{s+2} - \frac{4}{s+3}\right]$$

$$f(o^{+}) = \lim_{s\to\infty} sF(s)$$

$$f(o^{+}) = \lim_{s\to\infty} s\left[\frac{10}{s+2} - \frac{4}{s+3}\right]$$

$$f(o^{+}) = \lim_{s\to\infty} \frac{s}{s}\left[\frac{10}{1+2/s} - \frac{4}{1+3/s}\right]$$

$$f(o^{+}) = 10 - 4 = 6$$

Problem5: **Find the initial value of $\mathbf{f(t) = 2e^{-3t}}$**

Solution

$$F(s) = \frac{2}{s+3}$$

$$sF(s) = \frac{2s}{(s+3)}$$

$$f(o^+) = \lim_{s\to\infty} sF(s)$$

$$f(o^+) = \lim_{s\to\infty} \frac{2s}{(s+3)}$$

$$f(o^+) = \lim_{s\to\infty} \frac{2s}{(1+3/s)}$$

$$f(o^+) = 2$$

Final Value Theorem

- It allows to find the final values $f(\infty)$ from its Laplace transform and is given by

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Proof

Consider the Laplace transform of the differentiation function

$$L\left[\frac{d}{dt}(f(t))\right] = sF(s) - f(0)$$

Taking limit as $s \rightarrow 0$ on both sides

$$\lim_{s\to 0} L\left[\frac{d}{dt}(f(t))\right] = \lim_{s\to 0}[sF(s) - f(0)]$$

$$\lim_{s\to 0}\int_0^\infty e^{-st}\frac{d}{dt}(f(t))\,dt = \lim_{s\to 0}[sF(s) - f(0)]$$

$$\int_0^\infty e^{-(0)}\frac{d}{dt}(f(t))\,dt = \lim_{s\to 0}[sF(s)] - f(0)$$

$$\int_0^\infty 1.\frac{d}{dt}(f(t))\,dt = \lim_{s\to 0}[sF(s)] - f(0)$$

$$f(t)\Big|_0^{\infty} = \lim_{s\to 0}[sF(s)] - f(0)$$

$$f(\infty) - \cancel{(0)} = \lim_{s\to 0}[sF(s)] - \cancel{f(0)}$$

$$f(\infty) = \lim_{s\to 0}[sF(s)]$$

$$\boxed{f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)]}$$

Problem 1: **Find the final value of** $F(s) = \frac{s+6}{s(s+3)}$

Solution

$$sF(s) = \frac{(s+6)}{s\,(s+3)}$$

$$f(\infty) = \lim_{s\to 0} s(s)$$

$$f(\infty) = \lim_{s\to 0} \frac{(s+6)}{s\,(s+3)}$$

$$f(\infty) = \frac{6}{3} = 2$$

Problem 2: Find the final value of $F(s) = \frac{s+5}{s(s+1)(s+3)}$

Solution

$$sF(s) = \frac{s(s+5)}{(s+1)(s+3)}$$

$$f(\infty) = \lim_{s\to 0} s(s)$$

$$f(\infty) = \lim_{s\to 0} \frac{s(s+5)}{(s+1)(s+3)}$$

$$f(\infty) = \frac{5}{3}$$

Problem 3: **Find the final value of** $F(s) = \dfrac{\mathbf{0.32}}{(\boldsymbol{s}^2+\mathbf{2.4}\boldsymbol{s}+\mathbf{0.672})\boldsymbol{s}}$

Solution

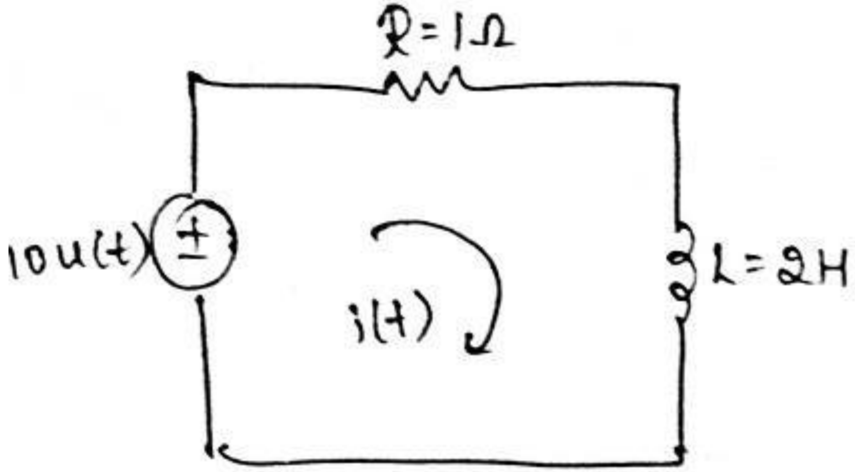
$$sF(s) = \frac{\mathrm{s} * 0.32}{(\mathrm{s}^2+2.4\mathrm{s} \ + 0.672)\mathrm{s}}$$

$$f(\infty) = \lim_{s \to 0} s(s)$$

$$f(\infty) = \lim_{s \to 0} \frac{\mathrm{s} * 0.32}{(\mathrm{s}^2+2.4\mathrm{s} + 0.672)\mathrm{s}}$$

$$f(\infty) = \frac{0.32}{0.672} = 0.476$$

Problem 4: Find the initial and final value of circuit given. (take I (0) = 1A)



Solution

Apply KVL to the loop

$$10\,u(t)\ -1\ * i(t) - 2\,\frac{di_{(t)}}{dt} = 0$$

$$10\,u(t) = 1\ * i(t) + 2\,\frac{di_{(t)}}{dt}$$

Take Laplace Transform on both sides

$$\frac{10}{s} = I(s) + 2\,[sI(s)\ - I(0)]$$

$$\frac{10}{s} = I(s) + 2\,[sI(s)\ - 1]$$

$$\frac{10}{s} = I(s) + 2sI(s)\ - 2$$

$$\frac{10}{s} + 2\ = I(s)(1+2s)$$

$$I(s) = \left(\frac{10}{s}+2\right)\left(\frac{1}{1+2s}\right)$$

$$sI(s) = s\left(\frac{10}{s}+2\right)\left(\frac{1}{1+2s}\right)$$

Initial Value

$$f(o^+) = \lim_{s\to\infty} sF(s)$$

$$i(o^+) = \lim_{s\to\infty} sI(s) = \lim_{s\to\infty} s\left(\frac{10}{s}+2\right)\left(\frac{1}{1+2s}\right)$$

$$i(o^+) = \lim_{s\to\infty} sI(s) = \lim_{s\to\infty} s\left(\frac{10}{s}+2\right)\left(\frac{1}{\frac{1}{S}+2}\right)\left(\frac{1}{S}\right)$$

$$i(o^+) = \frac{2}{2} = 1$$

Final Value

$$i(\infty) = \lim_{s\to 0} s(s)$$

$$i(\infty) = \lim_{s\to 0} s\left(\frac{10}{s} + 2\right)\left(\frac{1}{1+2s}\right)$$

$$i(\infty) = \lim_{s\to 0} s\left(\frac{(10+2S)}{S}\right)\left(\frac{1}{1+2s}\right)$$

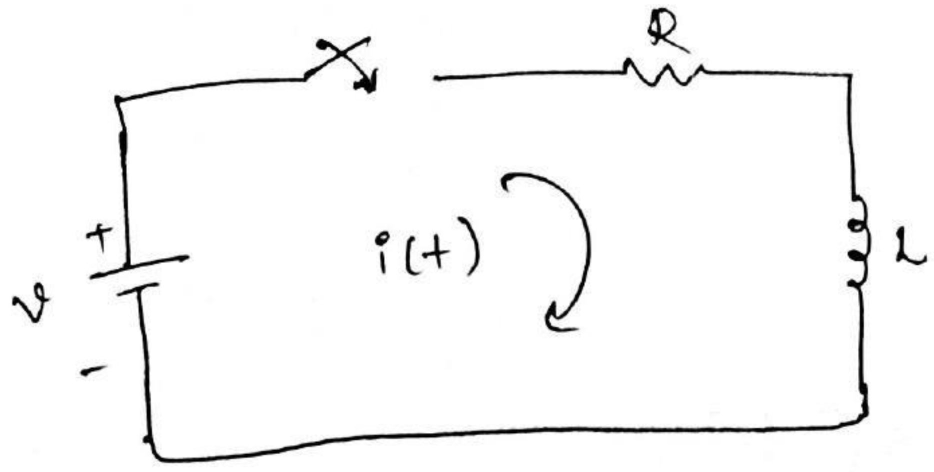
$$i(\infty) = 10$$

Network Analysis Using Laplace Transform

Elements	Time Domain	Laplace Domain
Resistor	$V(t) = I(t)\,R$	$V(s) = I(s)\,R$
Inductor	$v_L(t) = \dfrac{Ldi(t)}{dt}$	$V_L(s) = L[sI(s) - i(0)]$
Capacitor	$v_c(t) = \dfrac{1}{C}\int_0^t i(t)dt + V_c(0)$	$V_C(s) = \frac{1}{C}\left[\frac{I(s)}{s}\right] + \frac{V_c(0)}{s}$

Solve for Circuit Quantities Using Laplace Transform

Problem 1: In the circuit shown, the switch is closed at t = 0, calculate the expression of the resulting current.



Take Laplace Transform on both sides

$$\frac{v}{s} = RI(s) + L\,[sI(s) - I(0)]$$

$$\frac{v}{s} = RI(s) + L\,[sI(s) - 0]$$

$$\frac{v}{s} = I(s)[R + s\,L]$$

$$I(s) = \frac{v}{s(R + sL)}$$

$$I(s) = \frac{v/L}{s(R/L + s)}$$

Using partial fractions

$$I(s) = \frac{A}{s} + \frac{B}{\frac{R}{L} + s} \quad \ldots\ldots e(1)$$

Solution

$$i(0^-) = 0\ A$$

Apply KVL to the loop

$$v - R * i(t) - L\frac{di(t)}{dt} = 0$$

$$v = R * i(t) + L\frac{di(t)}{dt}$$

$$I_{(s)} = \frac{A}{s} + \frac{B}{\frac{R}{L} + s} \; \ldots\ldots e(1)$$

$$\frac{\frac{v}{L}}{s\left(\frac{R}{L} + s\right)} = \frac{A\left(s + \frac{R}{L}\right) + B_{(s)}}{s\left(s + \frac{R}{L}\right)}$$

$$\frac{v}{L} = A\left(s + \frac{R}{L}\right) + B(s) \; \ldots\ldots e(2)$$

Put s =0 in eq (2)

$$\frac{v}{L} = A\left(0 + \frac{R}{L}\right) + (0)$$

$$A = \frac{v}{R}$$

Put $s = -\frac{R}{L}$ in eq (2)

$$\frac{v}{L} = A\left(-\frac{R}{L} + \frac{R}{L}\right) + B\left(-\frac{R}{L}\right)$$

$$\frac{v}{L} = 0 + B\left(-\frac{R}{L}\right)$$

$$B = \frac{-v}{R}$$

Substitute the value of A and B in eq (1)

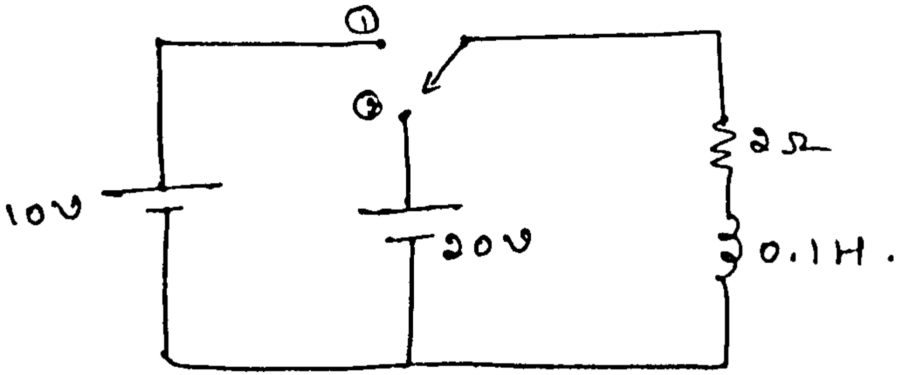
$$I(s) = \frac{v/R}{s} - \frac{v/R}{\frac{R}{L} + s}$$

Take Inverse Laplace Transform on both sides

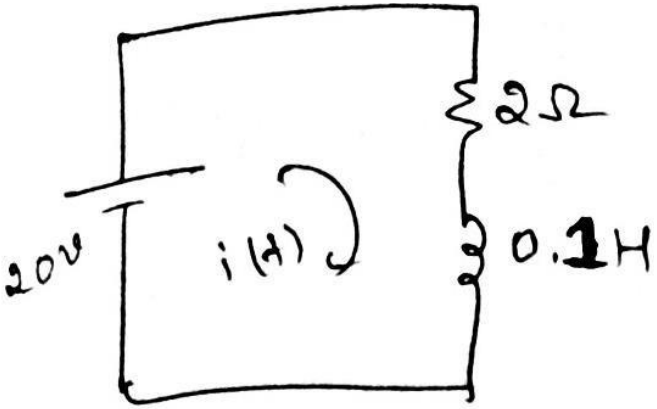
$$i(t) = \left(\frac{\mathrm{v}}{\mathrm{R}} - \frac{\mathrm{v}}{\mathrm{R}}\, \mathrm{e}^{-\left(\frac{\mathrm{R}}{\mathrm{L}}\right)\mathrm{t}}\right) \mathrm{u(t)}$$

$$i(t) = \frac{\mathrm{v}}{\mathrm{R}}\left(1 - \mathrm{e}^{-\left(\frac{\mathrm{R}}{\mathrm{L}}\right)\mathrm{t}}\right) u(t)$$

Problem 2: **For the circuit given, determine current when the switch is moved from position 1 to position 2 at t=0**

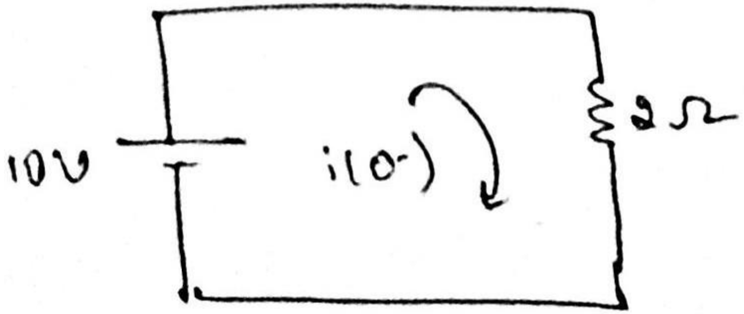


At t > 0



Solution

When the switch is at position 1 i.e., at $t = 0^-$



$$i(0^-) = \frac{10}{2} = 5\ A$$

Apply KVL to the loop

$$20 - 2 * i(t) - 0.1\ \frac{di(t)}{dt} = 0$$

Take Laplace Transform on both sides

$$\frac{20}{s} = 2I(s) + 0.1\ [sI(s) - I(0)]$$

$$\frac{20}{s} = 2I(s) + 0.1\ [sI(s) - 5]$$

$$\frac{20}{s} = 2I(s) + 0.1\,[sI(s)] - 0.5$$

$$\frac{20}{s} + 0.5 = I(s)[2 + 0.1\,s]$$

$$I(s) = \left(\frac{20}{s} + 0.5\right)\left(\frac{1}{2 + 0.1\,s}\right)$$

$$I(s) = \left(\frac{20 + 0.5s}{s}\right)\left(\frac{1}{2 + 0.1\,s}\right)$$

$$I(s) = \frac{0.5(40 + s)}{0.1(20 + s)}$$

$$I(s) = \frac{5(40 + s)}{(20 + s)}$$

Using partial fractions

$$I(s) = \frac{A}{s} + \frac{B}{20 + s} \quad \dots\dots \mathrm{eq}(1)$$

$$\frac{5(40 + s)}{(20 + s)} = \frac{A}{s} + \frac{B}{20 + s}$$

$$5(40 + s) = (20 + s) + B(s) \dots \mathrm{eq}(2)$$

Put s =0 in eq (2)

$$5(40 + 0) = (20 + 0) + B(0)$$

$$A = 10$$

Put $s = -20$ in eq (2)

$$5(40 - 20) = (20 - 20) + B(-20)$$

$$B = -5$$

Substitute the value of A and B in eq (1)

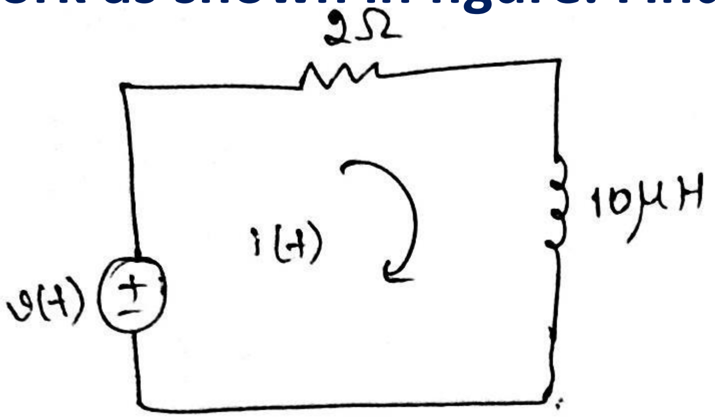
$$I(s) = \frac{10}{s} + \frac{-5}{20 + s}$$

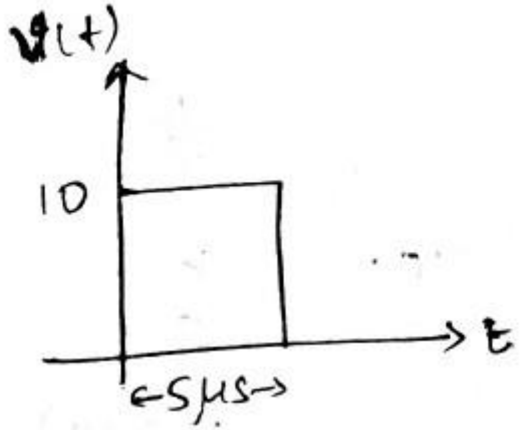
Take Inverse Laplace Transform on both sides

$$i(t) = \left(10 - 5e^{(-20)t}\right)u(t)$$

$$i(t) = 5(2 - e^{-20t})\,u(t)$$

Problem 3: **A pulse voltage of 10 V magnitude and 5μs duration is applied to the RL network as shown in figure. Find i(t)**





Solution

$$v(t) = 10u(t) - 10u(t-5\mu)$$

Apply KVL to the loop

$$v(t) - 2 * i(t) - 10\mu \frac{di(t)}{dt} = 0$$

$$10u(t) - 10u(t-5\mu) = 2 * i(t) + 10\mu \frac{di(t)}{dt}$$

Take Laplace Transform on both sides

$$\frac{10}{s} - \frac{10e^{-5\mu s}}{s} = 2I(s) + 10\mu[sI(s) - I(0)]$$

$$\frac{10}{s} - \frac{10e^{-5\mu s}}{s} = 2I(s) + 10\mu[sI(s) - 0]$$

$$\frac{10}{s}(1 - e^{-5\mu s}) = I(s)[2 + 10\mu s]$$

$$I(s) = \frac{10}{s}\frac{(1 - e^{-5\mu s})}{(2 + 10\mu s)}$$

$$I(s) = \frac{10}{s}\frac{(1-e^{-5\mu s})}{(2+10\mu s)}$$

$$I(s) = \frac{10}{s}\frac{(1-e^{-5\mu s})}{10(\frac{2}{10\mu}+s)}$$

$$I(s) = \frac{10^6}{s}\frac{(1-e^{-5\mu s})}{(\frac{10^6}{5}+s)}$$

$$I(s) = 10^6\left(\frac{1}{s\left(\frac{10^6}{5}+s\right)}\right)(1-e^{-5\mu s})$$

$$I(s) = A(S)(1-e^{-5\mu s})$$

$$A(S) = 10^6\frac{1}{s\left(\frac{10^6}{5}+s\right)}$$

Solve for A(s) using partial fractions

$$A(S) = \frac{A}{s} + \frac{B}{\frac{10^6}{5}+s} \quad \ldots e(1)$$

$$10^6\frac{1}{s\left(\frac{10^6}{5}+s\right)} = \frac{A}{s} + \frac{B}{\frac{10^6}{5}+s}$$

$$10^6 = A\left(\frac{10^6}{5}+s\right) + B(s) \quad \ldots \text{eq}(2)$$

Put s =0 in eq (2)

$$10^6 = A\left(\frac{10^6}{5}+0\right) + B(0)$$

$$A = 5$$

Put $s = -\frac{10^6}{5}$in eq (2)

$$10^6 = A\left(\frac{10^6}{5} - \frac{10^6}{5}\right) + B\left(-\frac{10^6}{5}\right)$$

$$B = -5$$

Substitute the value of A and B in eq (1)

$$A(S) = \frac{5}{s} + \frac{-5}{\frac{10^6}{5} + s}$$

Substitute the value of A(s) in the equation for I(s)

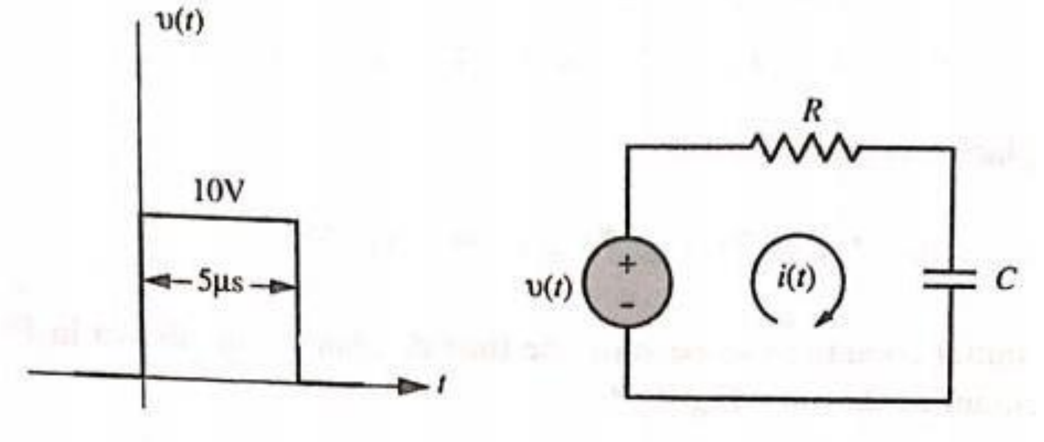
$$I(s) = \left(\frac{5}{s} + \frac{-5}{\frac{10^6}{5} + s}\right)(1 - e^{-5\mu s})$$

$$I(s) = \frac{5}{s} - \frac{5}{s}e^{-5\mu s} - \frac{5}{s + \frac{10^6}{5}} + \frac{5}{s + \frac{10^6}{5}}e^{-5\mu s}$$

Take Inverse Laplace Transform on both sides

$$i(t) = 5u(t) - 5u(t - 5\mu) - 5u(t)e^{-\frac{10^6}{5}t} + 5u(t - 5\mu)e^{-\frac{10^6(t - 5\mu)}{5}}$$

Problem 4: **A voltage pulse of 10V magnitude and 5µs duration is applied to the RC network shown in figure. Find i(t) if R =10Ω and C = 0.05µF**



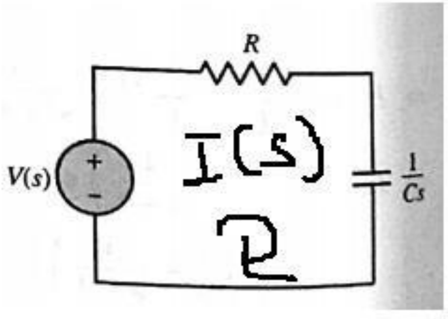
Solution

$$v(t) = 10u(t) - 10u(t - 5\mu)$$

$$V(s) = \frac{10}{s} - \frac{10}{s}\, e^{-5\mu s}$$

$$V(s) = \frac{10}{s}(1 - e^{-5\mu s})$$

Transform the circuit to s domain



$$I(s) = \frac{V(s)}{R + \frac{1}{Cs}}$$

$$I(s) = \frac{\frac{10}{s}(1 - e^{-5\mu s})}{R + \frac{1}{Cs}}$$

$$I(s) = \frac{\frac{10}{s}(1 - e^{-5\mu s})}{\frac{RCs + 1}{Cs}}$$

$$I(s) = \frac{10Cs(1 - e^{-5\mu s})}{(RCs + 1)}$$

$$I(s) = \frac{10C(1 - e^{-5\mu s})}{(RCs + 1)}$$

Divide both numerator and denominator by RC

$$I(s) = \frac{\frac{10}{R}(1 - e^{-5\mu s})}{s + \frac{1}{RC}}$$

$$I(s) = \frac{10}{R}\left[\frac{1}{s + \frac{1}{RC}} - \frac{1}{s + \frac{1}{RC}} e^{-5\mu s}\right]$$

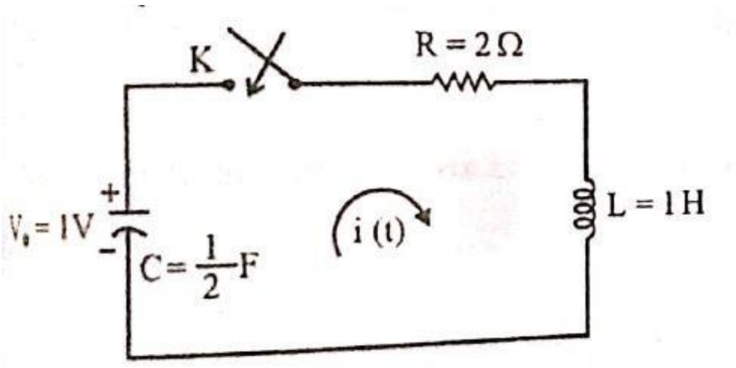
Take Inverse Laplace Transform on both sides

$$i(t) = \frac{10}{R} u(t) e^{-\frac{1}{RC}t} - \frac{10}{R} u(t - 5\mu) e^{-\frac{1(t-5\mu)}{RC}}$$

Substitute R =10Ω and C = 0.05µF

$$i(t) = u(t)e^{-2 * 10^6 t} - u(t - 5\mu)e^{-2 * 10^6 (t - 5\mu)}$$

Problem 5: **In the circuit shown in figure, if the capacitor is initially charged to 1V, find an expression for i(t), when the switch K is closed at t = 0**



Solution

Apply KVL to the loop

$$-\frac{1}{C}\int i(t)dt + v(0) - 2\, i(t) - L\frac{di(t)}{dt} = 0$$

$$\frac{1}{C}\int i(t)dt - v(0) + 2\, i(t) + L\frac{di(t)}{dt} = 0$$

Take Laplace Transform on both sides

$$\frac{1}{Cs} I(s) - \frac{v(o)}{s} + 2\, I(s) + Ls\, I(s) = 0$$

$$\left(\frac{1}{Cs} + 2 + Ls\right) I(s) = \frac{v(o)}{s}$$

$$I(s) = \frac{\frac{v(o)}{s}}{\frac{1}{Cs} + 2 + \text{Ls}}$$

$$I(s) = \frac{\frac{1}{s}}{\frac{2}{s} + 2 + \text{s}}$$

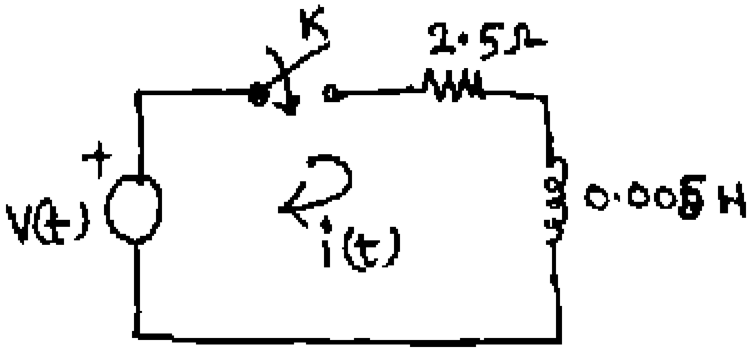
$$I(s) = \frac{\frac{1}{s}}{\frac{2 + 2s + s^2}{s}}$$

$$I(s) = \frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1}$$

Take Inverse Laplace Transform on both sides

$$i(t) = \sin(t)\, e^{-t}$$

Problem 6: **In series RL circuit shown, the source voltage $v(t) = 50sin250t\ V$. Using Laplace Transform determine the current when the switch k is closed at t = 0**

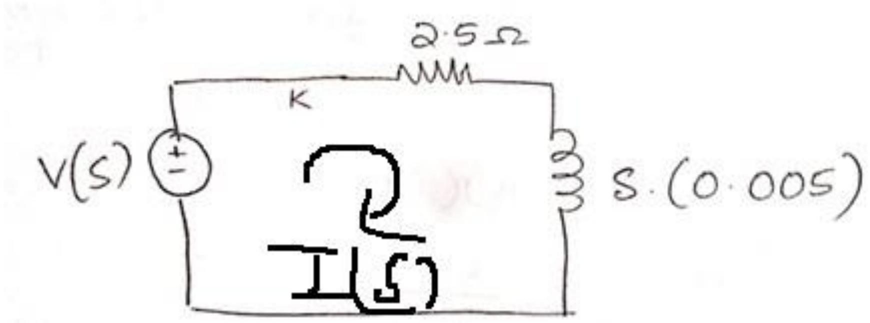


Solution

$$v(t) = 50\sin 250\ t$$

$$V(s) = 50\ * \frac{250}{s^2 + 250^2}$$

Transform the circuit to s domain



$$I(s) = \frac{V(s)}{R + sL}$$

$$I(s) = \frac{50\ * \frac{250}{s^2 + 250^2}}{2.5\ + (0.005)}$$

$$I(s) = \frac{12500}{(s^2+250^2)\ (2.5\ + (0.005))}$$

$$I(s) = \frac{12500}{(s^2+250^2)\ (\frac{2.5}{0.005}\ + s)0.005}$$

$$I(s) = \frac{12500}{(s^2+250^2)\ (500 + s)0.005}$$

$$I(s) = \frac{2.5 * 10^6}{(s^2+250^2)\ (500 + s)}$$

Using partial fractions

$$I(s) = \frac{A}{s+500} + \frac{Bs+C}{s^2+250^2} \quad \ldots eq(1)$$

$$\frac{2.5 * 10^6}{(s^2+250^2)\ (500 + s)} = \frac{A}{s+500} + \frac{Bs+C}{s^2+250^2}$$

$$2.5 * 10^6 = A(s^2 + 250^2) + (Bs + C)(s + 500) \ldots eq(2)$$

Put s =-500 in eq (2)

$$2.5 * 10^6 = A\left((-500)^2+250^2\right) + (Bs + C)(-500 + 500) \ldots eq(2)$$

$$A = 8$$

Multiply, separate the terms and equate

$$2.5 * 10^6 = A(s^2) + A(250^2) + (Bs^2 + Bs(500) + Cs + (C * 500))$$

$$2.5 * 10^6 = (A + B)s^2 + s(500B + C) + (A * 250^2 + C * 500)$$

$$0 = A + B \quad \ldots eq(3)$$

$$0 = 500B + C \ldots eq(4)$$

$$2.5 * 10^6 = A * 250^2 + C * 500 \ldots eq(5)$$

From eq(3)

$$B = -A = -8$$

From eq(4)

$$C = -500B = -500(-8) = 4000$$

Substitute the value of A, B and C in eq (1)

$$I(s) = \frac{8}{s+500} + \frac{-8s+4000}{s^2+250^2}$$

$$I(s) = \frac{8}{s+500} - \frac{8s}{s^2+250^2} + \frac{4000}{s^2+250^2}$$

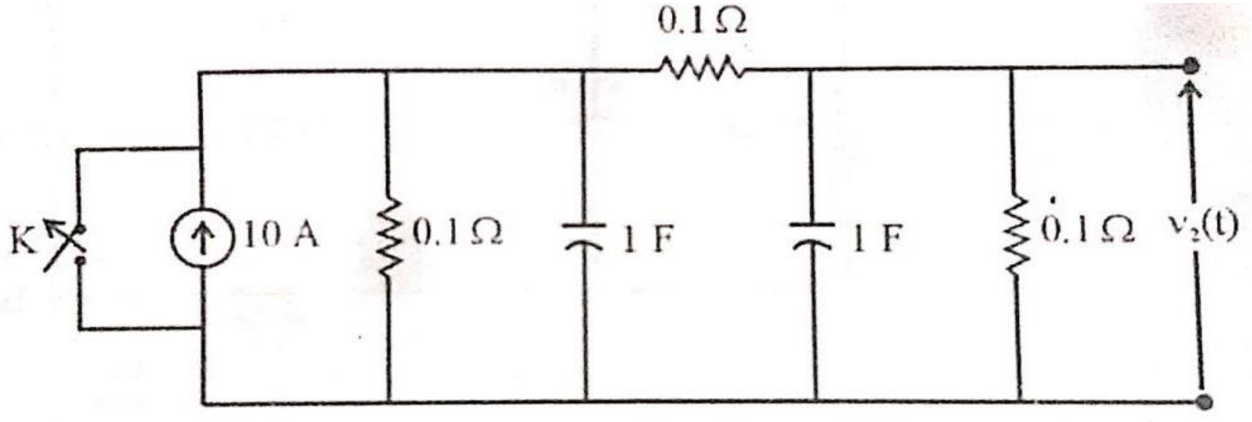
$$I(s) = \frac{8}{s+500} - \frac{8s}{s^2+250^2} + \frac{4000}{250}\left(\frac{250}{s^2+250^2}\right)$$

$$I(s) = \frac{8}{s+500} - \frac{8s}{s^2+250^2} + 16\left(\frac{250}{s^2+250^2}\right)$$

Take Inverse Laplace Transform on both sides

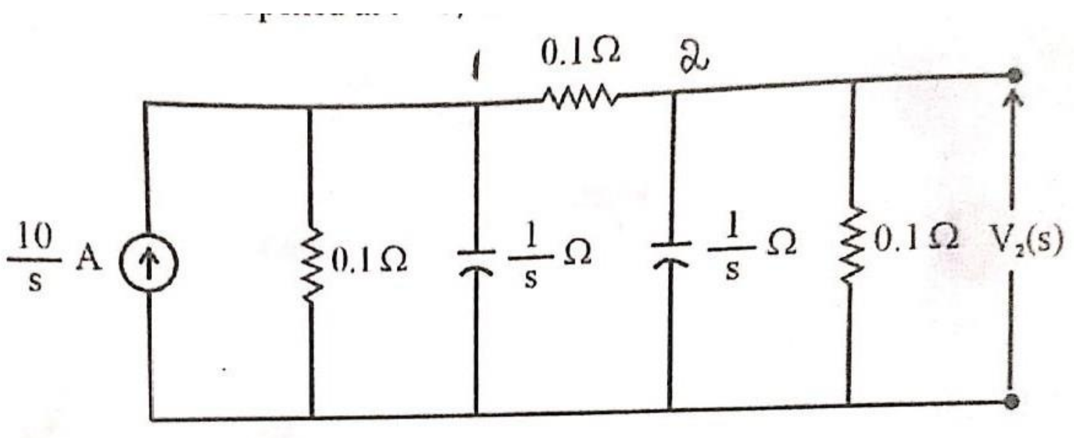
$$(t) = 8u(t)e^{-500t} - 8\cos(250t) + 16\sin(250t)$$

Problem 7: For the circuit shown in figure, find the voltage $V_2(t)$, when the switch is opened at t =0. Assume all the initial conditions to be zero.



Solution

Transform the circuit to s domain



Apply KCL at node 1

$$\frac{V_1(s)}{\frac{1}{s}} + \frac{V_1(s)}{0.1} - \frac{10}{s} + \frac{V_1(s) - V_2(s)}{0.1} = 0$$

$$sV_1(s) + 10V_1(s) + 10V_1(s) - 10V_2(s) = \frac{10}{s}$$

$$V_1(s)(s+20) - 10V_2(s) = \frac{10}{s} \quad \ldots .e(1)$$

Apply KCL at node 2

$$\frac{V_2(s)}{\frac{1}{s}} + \frac{V_2(s)}{0.1} + \frac{V_2(s) - V_1(s)}{0.1} = 0$$

$$sV_2(s) + 10V_2(s) + 10V_2(s) - 10V_1(s) = 0$$

$$-10V_1(s) + V_2(s)(s+20) = 0 \ldots .e(2)$$

Solve for using V_2 Cramer's rule

$$\Delta = \begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}$$

$$\Delta = (s+20)^2 - 100$$

$$\Delta = S^2 + 40s + 400 - 100$$

$$\Delta = S^2 + 40s + 300$$

$$\Delta_2 = \begin{vmatrix} s+20 & \frac{10}{s} \\ -10 & 0 \end{vmatrix}$$

$$\Delta_2 = 0 + \frac{100}{s}$$

$$\Delta_2 = \frac{100}{s}$$

$$V_2(s) = \frac{\Delta_2}{\Delta} = \frac{\frac{100}{s}}{s^2 + 40s + 300}$$

$$V_2(s) = \frac{100}{(s^2 + 40s + 300)} = \frac{100}{(s+30)(s+10)}$$

Using partial fractions

$$V_2(s) = \frac{A}{s} + \frac{B}{s+30} + \frac{C}{s+10} \quad ..\text{eq}(3)$$

$$\frac{100}{(s+30)(s+10)} = \frac{A}{s} + \frac{B}{s+30} + \frac{C}{s+10}$$

$$100 = A(s+30)(s+10) + Bs(s+10) + Cs(s+30)..e(4)$$

Put s =0 in eq (4)

$$100 = A(0+30)(0+10) + 0 + 0$$

$$A = \frac{1}{3}$$

Put s =-30 in eq (4)

$$100 = 0 + (-30) \ (-30+10) + 0$$

$$B = \frac{1}{6}$$

Put s =-10 in eq (4)

$$100 = 0 + 0 + (-10) \quad (-10 + 30)$$

$$C = \frac{-1}{2}$$

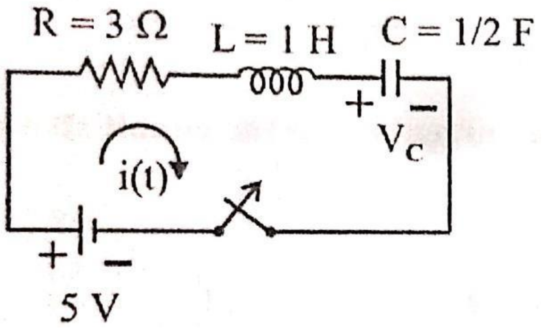
Substitute the value of A , B and C in eq (3)

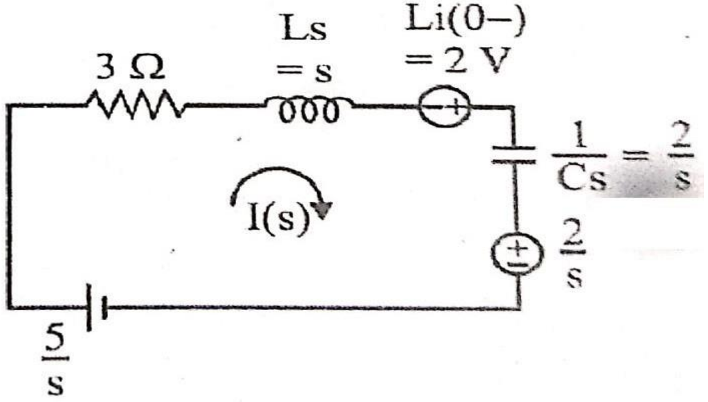
$$V_2(s) = \frac{1}{3s} + \frac{1}{6(s+30)} - \frac{1}{2(s+10)}$$

Take Inverse Laplace Transform on both sides

$$V_2(t) = \frac{1}{3}u(t) + \frac{1}{6}u(t)e^{-30t} - \frac{1}{2}u(t)\,e^{-10t}$$

Problem 8: For the series RLC circuit shown in figure, the initial conditions are $i_{LO} = 2\ A\ and\ V_{CO} = 2\ V$. It is connected to a DC voltage at t=0. Find the current i(t) after the switching action, using Laplace Transform.





Solution

Transform the circuit to s domain

$$Ls = 1 * s = s$$

$$Li(0^-) = 1 * 2 = 2\text{V}$$

$$\frac{1}{Cs} = \frac{1}{\left(\frac{1}{2}\right)s} = \frac{2}{s}$$

$$\frac{V_c(o^-)}{s} = \frac{2}{s}$$

$$I(s) = \frac{\frac{5}{s} + 2 - \frac{2}{s}}{3 + s + \frac{2}{s}}$$

$$I(s) = \frac{2s+3}{s^2+3s+2} = \frac{2s+3}{(s+1)(s+2)}$$

Using partial fractions

$$I(s) = \frac{A}{s+1} + \frac{B}{s+2} \ldots \text{eq}(1)$$

$$\frac{2s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$2s + 3 \ = A(s + 2) + B(s + 1) \ldots e(2)$$

Put s =-2 in eq (2)

$$2(-2) + 3 \ = 0 + B(-2 + 1) \ldots e(2)$$

$$A = 1$$

Put s =-1 in eq (2)

$$2(-1) + 3 \ = A(-1 + 2) + 0$$

$$B = 1$$

Substitute the value of A and B in eq (1)

$$I(s) = \frac{1}{s+1} + \frac{1}{s+2}$$

Take Inverse Laplace Transform on both sides

$$i(t) = u(t)e^{-t} + u(t)e^{-2t}$$

Assignment: Deduce the Laplace Transform of the following

1. $\boldsymbol{sin^2 t}$

2. $\boldsymbol{cos^2 t}$

3. $\boldsymbol{sinwt}$

4. $\boldsymbol{\int_0^t i(t)dt}$

MODULE-2 NETWORK THEOREMS

By
K PRABHAVATHI
Assistant Professor
Department of ECE
BGSIT, BG Nagara

Contents

Superposition Theorem

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Superposition Theorem

"If any multisource complex network consisting of linear bilateral elements, the voltage across or current through, any given element of the network is equal to the sum of the individual voltages or currents, produced independently across or in that element by each source acting independently, when all the remaining sources are replaced by their respective internal impedances.

If the internal impedances of the sources are unknown then the independent voltage sources must be replaced by short circuit while the independent current sources must be replaced by an open circuit.

If there are dependent current or voltage sources present in the circuit then such dependent sources should not be replaced by open or short circuit and must be kept as it is."

Steps to Apply Superposition Theorem

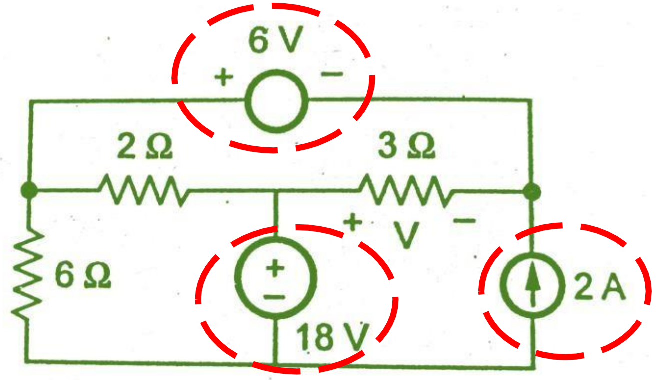
Step1: ***Select a single source acting alone. Short the other voltage sources and open the current sources, if internal impedances are not known. If known, replace them by their internal impedances.***

Step2: ***Find the current through or the voltage across the required element, due to the source under consideration, using a suitable network simplification technique.***

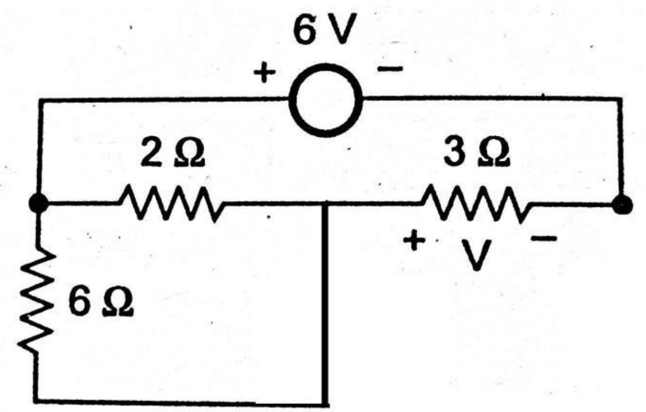
Step3: ***Repeat the above two steps for all the sources.***

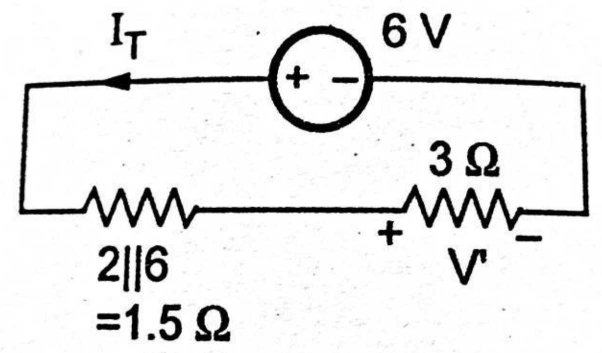
Step4: ***Add all the individual effects produced by individual sources, to obtain the total current in or voltage across the element.***

Problem1: Find the voltage 'V' across 3Ω resistor using superposition theorem for the circuit given.



Step1: Consider 6V alone, short 18V and open 2A.

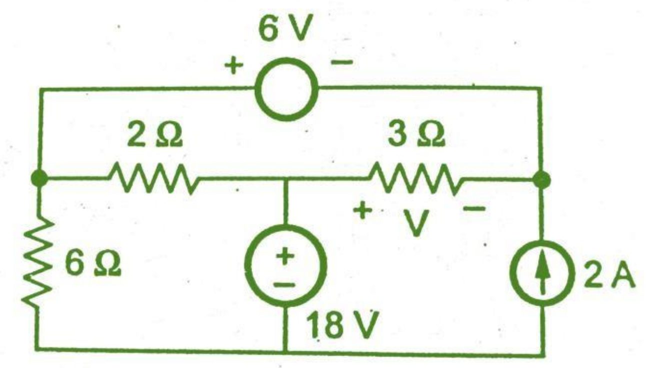


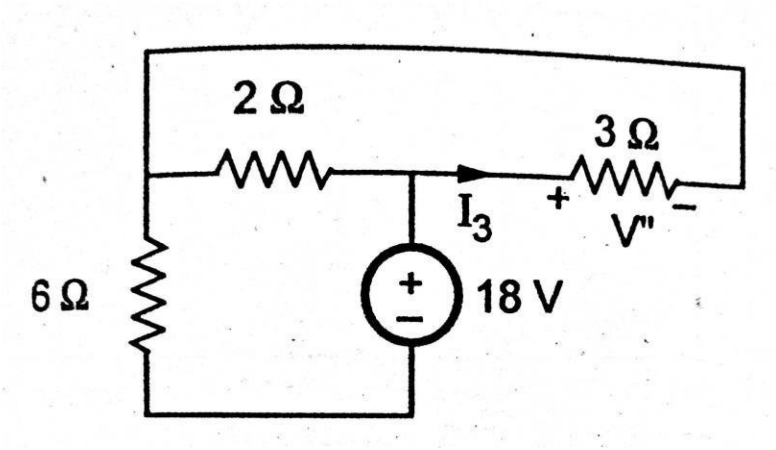


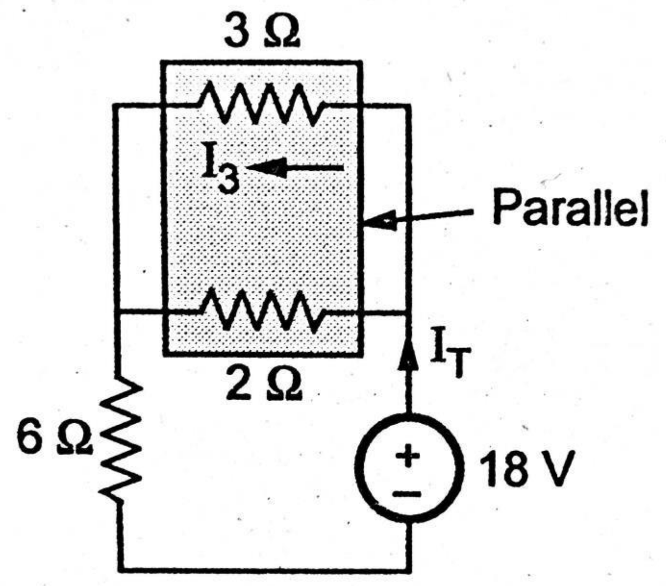
$$I_T = \frac{6}{1.5+3} = 1.33 \text{ A}$$

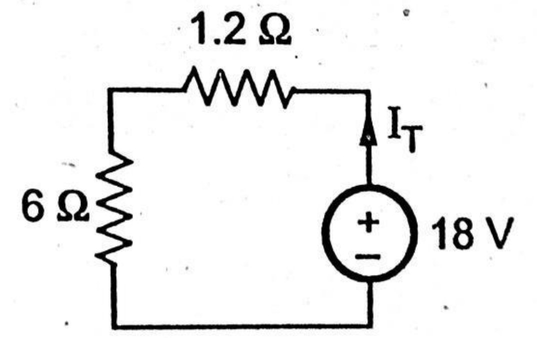
$$V' = 3\, I_T = 4\text{V}$$

Step2: Consider 18V alone, short 6V and open 2A.









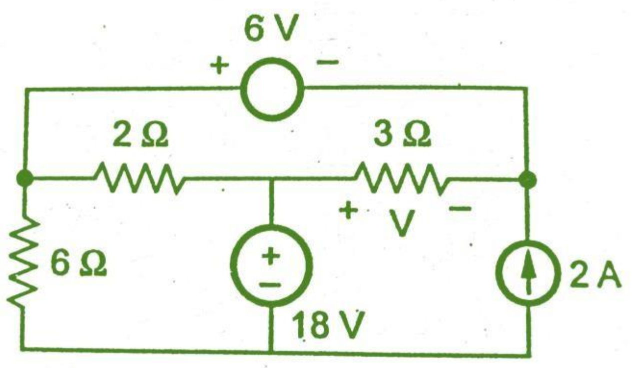
$$I_T = \frac{18}{6+1.2} = 2.5 \text{ A}$$

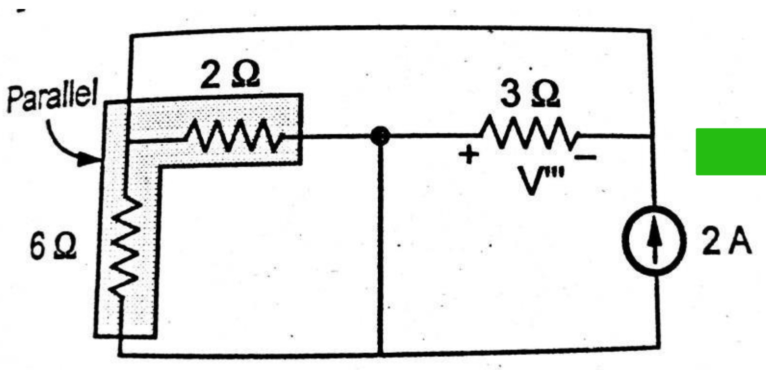
Applying Current division rule,

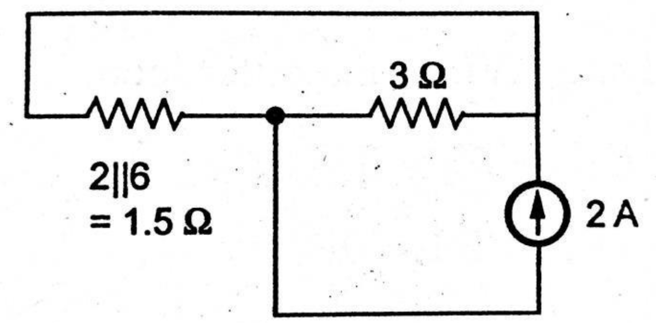
$$I_3 = I_T \times \frac{2}{2+3} = \frac{2.5X2}{5} = 1 \text{ A}$$

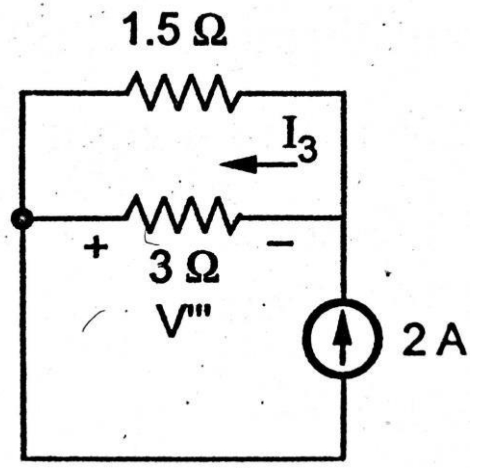
$$V'' = 3\, I_3 = 3 \text{ X } 1 = 3\text{V}$$

Step3: Consider 2A alone, short 6V and short 18V.









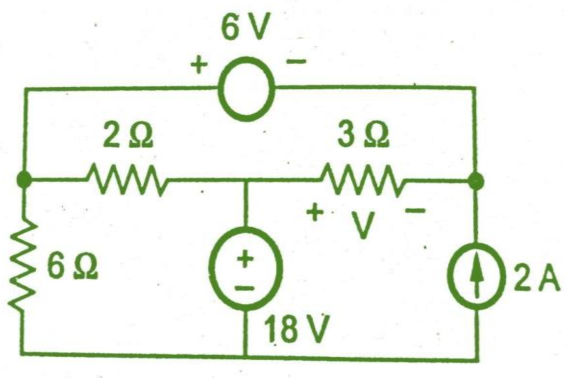
Applying Current division rule,

$$I_3 = 2 \times \frac{1.5}{3+1.5} = 0.666 \text{ A}$$

$$V''' = -3\, I_3 = -3 \times 0.666 = -2\text{V}$$

$$\text{V} = V' + V'' + V''' = 4+3-2 = 5\text{V}$$

VERIFICATION



I_2 = -2A

At loop1,

-6I_1-2I_1+2I_3-18=0

-8I_1+2I_3 = 18 ------------------(1)

At loop3,

-2I_3+ 2I_1-3I_3+3I_2-6 =0

2I_1+3I_2-5I_3 =6

2I_1+3 (-2) - 5I_3 =6

2I_1 - 5I_3 = 12 -------------------(2)

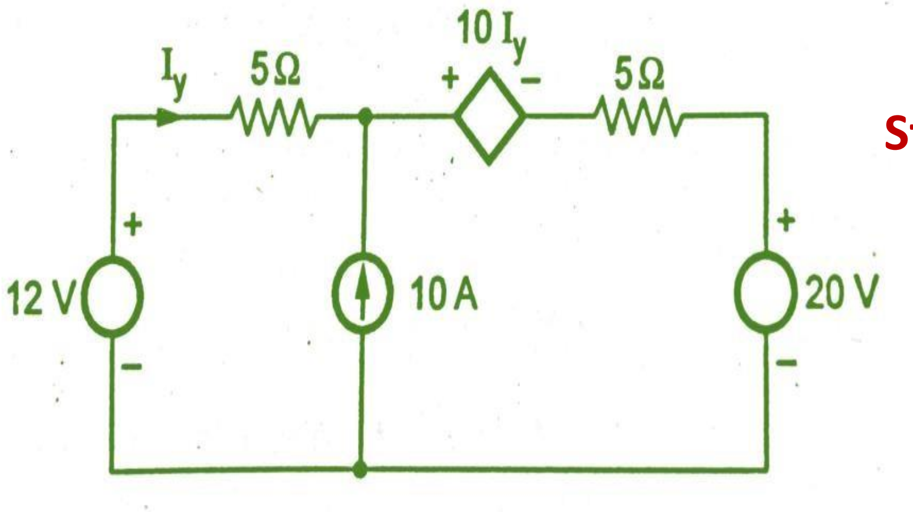
I_1= -3.166 A

I_3 = -3.667 A

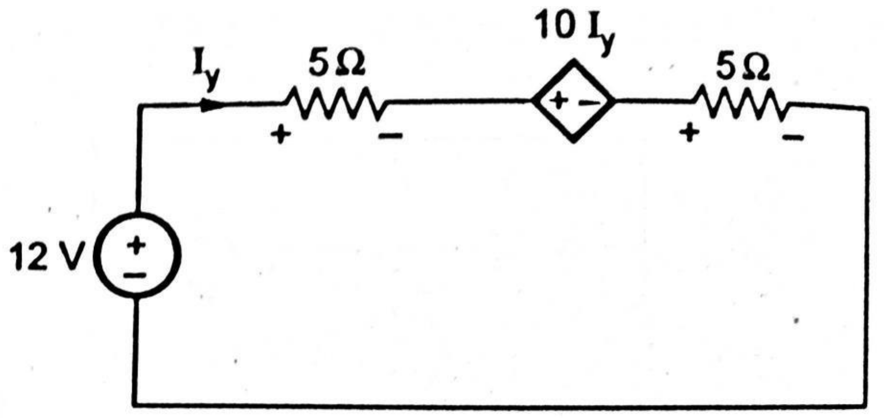
I_2-I_3 = -2+3.667 = 1.667A

V = 3 X $(I_3 - I_2)$ = 3X1.667 =5V

Problem2: Find the current in 5Ω resistors using superposition theorem.



Step1: Consider 12V only, short 20V and open 10A.



Applying KVL,

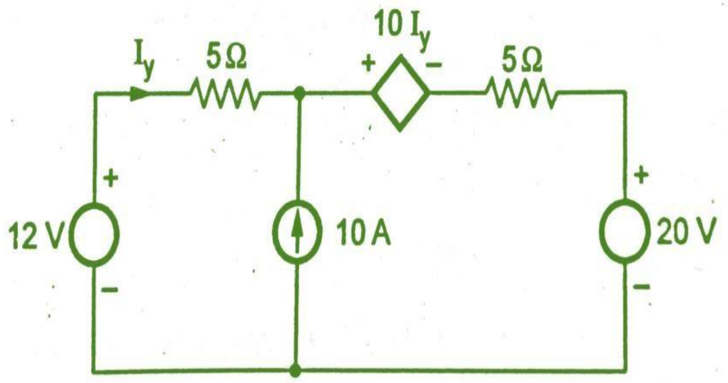
-5I_y - 10I_y -5I_y + 12 =0

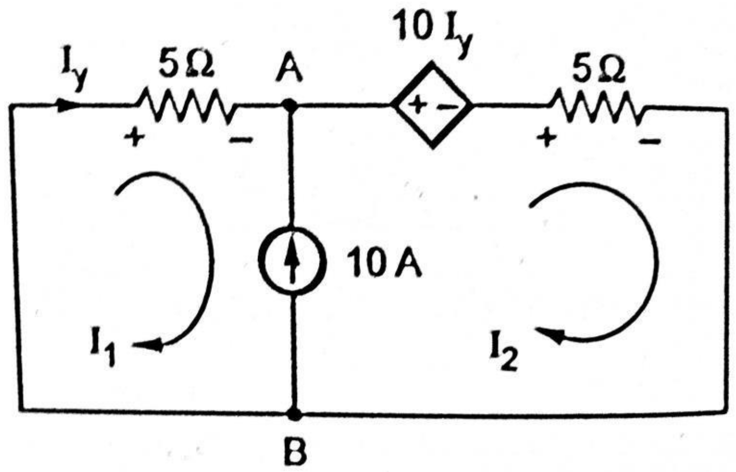
-20I_y = -12

I_y= 0.6 A

I_y= 0.6 A through both 5Ω resistances due to 12V.

Step2: **Consider 10A only, short 12V and 20V.**





From the branch AB,

I_2-I_1=10 --------------------------(1)

I_1= I_y

Applying KVL to the outer loop

-5I_1 - 10I_y -5I_2 =0

-5I_1 - 10I_1 -5I_2 =0

-15I_1-5I_2 = 0 -------------------(2)

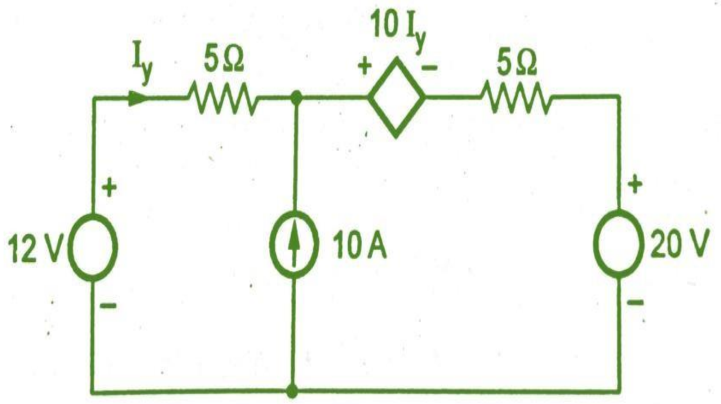
I_1= -2.5 A

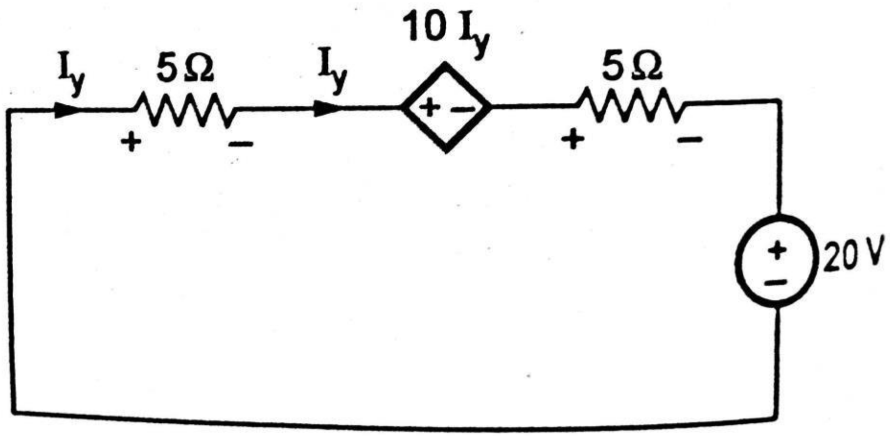
I_2= 7.5 A

The current through left 5Ω resistor is 2.5A due to 10 A

The current through right 5Ω resistor is 7.5A

Step3: Consider 20v only, short 12V and open 10A.





Applying KVL,

-5I_y - 10I_y -5I_y - 20 =0

-20I_y = 20

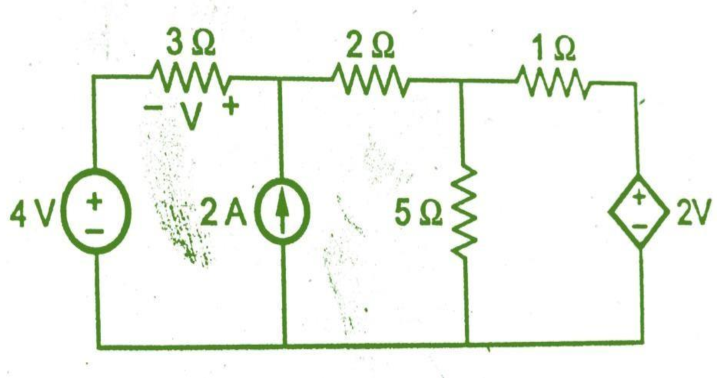
I_y= -1 A

I_y= 1 A through both 5Ω resistances due to 20V.

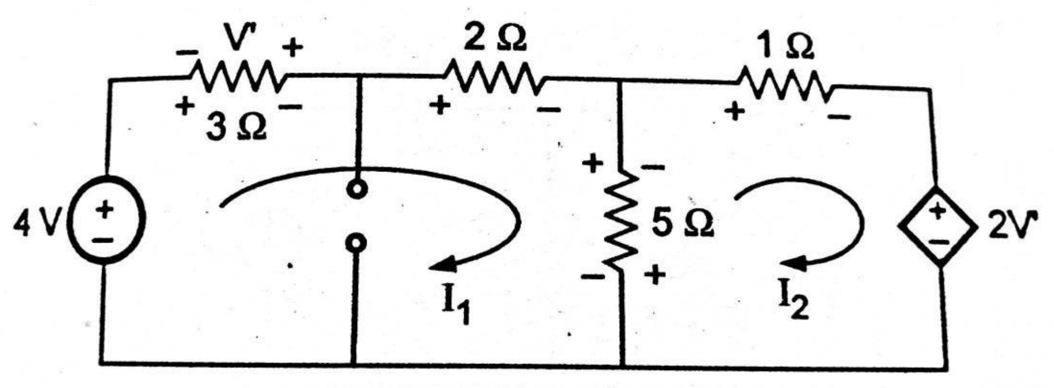
I = I′ + I″ + I‴ = 0.6 – 2.5 -1 = -2.9A

I = I′ + I″ + I‴ = 0.6 + 7.5 -1 = 7.1A

Problem3: **Find 'V' using superposition theorem in the network shown.**



Step1: **Consider 4V only and open 2A source.**



Applying KVL,

-3I_1 - 2I_1 -5I_1 + 5I_2+ 4 =0

-10I_1+ 5I_2 = -4 --------------------(1)

-I_2 - 2V' -5I_2 +5I_1= 0

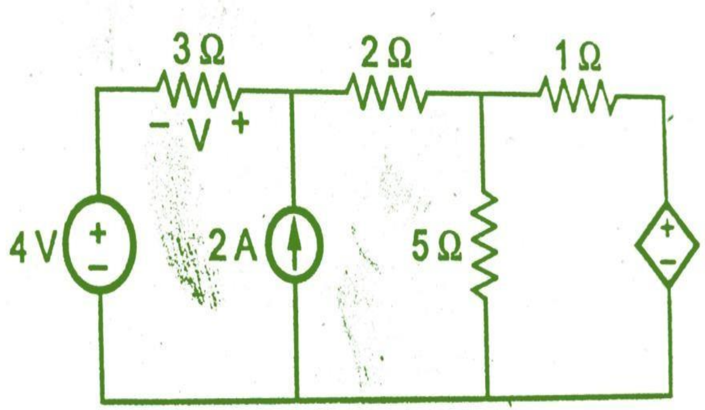
-I_2 - 2(-3I_1) -5I_2 +5I_1= 0

11I_1 -6I_2 = 0 --------------------(2)

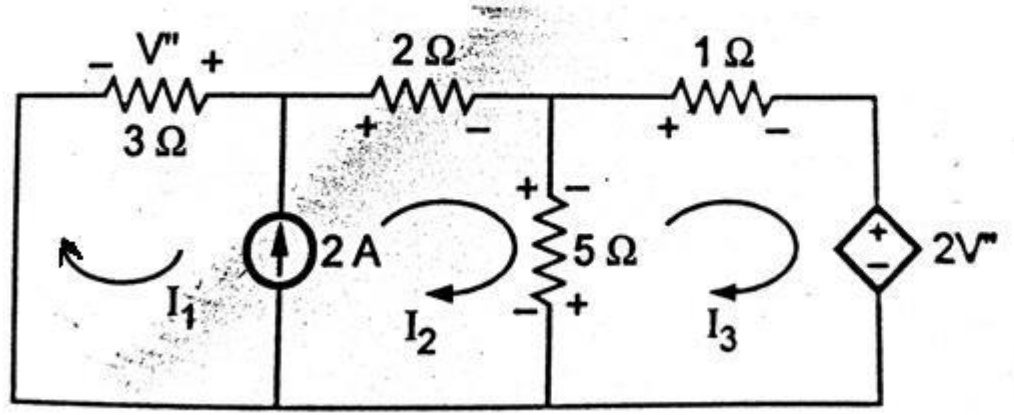
I_1= 4.8 A, **I_2= 8.8 A**

V' = -3I_1

V' = -14.4 V



tep2: Consider 2A only and short 4V source.



I_2-I_1 = 2 -------------------------(1)

Applying KVL

-3I_1 - 2I_2 -5I_2 +5I_3= 0

-3I_1 - 7I_2 +5I_3= 0 -------------------(2)

-I_3 -2V'' -5I_3 +5I_2= 0

-I_3 -2(-3I_1) -5I_3 +5I_2= 0

6I_1 + 5I_2 -6I_3= 0 --------------------(3)

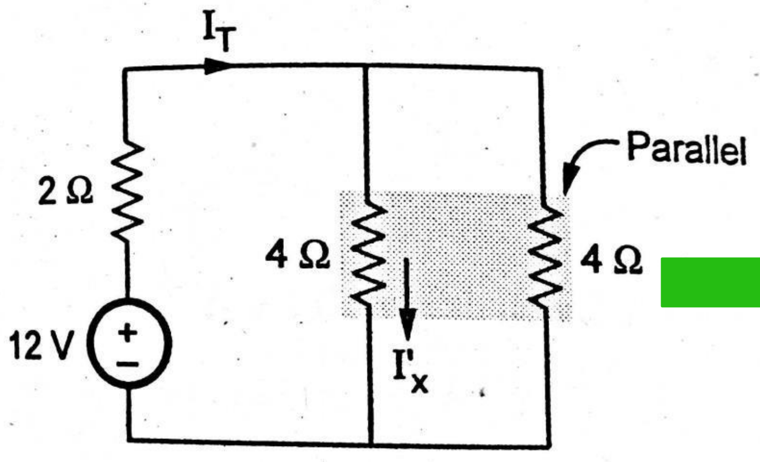
I_1= -6.8 A, I_2= -4.8 A, I_3= -10.8 A

V' = -3I_1

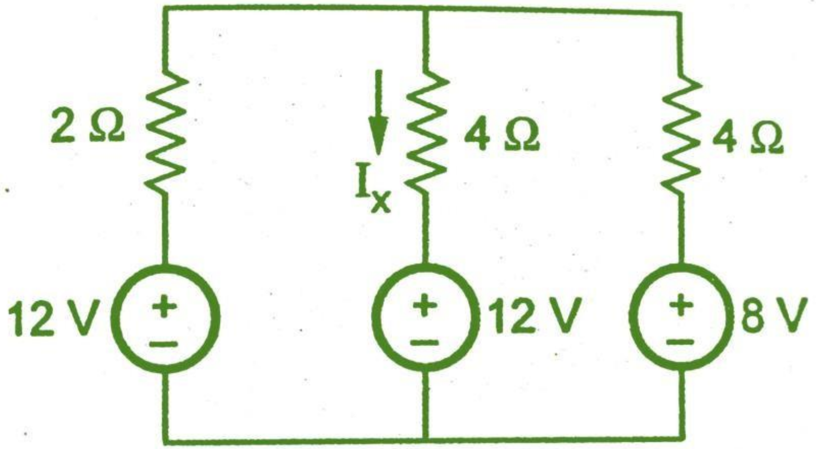
V' = 20.4 V

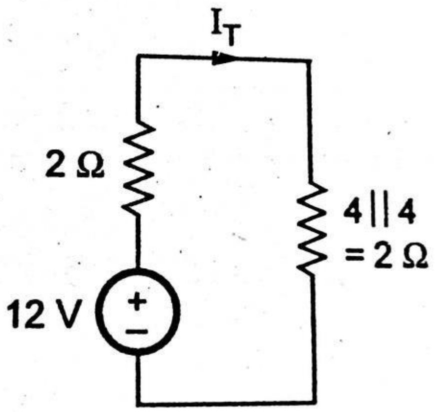
V = V' + V'' = -14.4 + 20.4 = 6V

Problem4: Find 'I_x' using superposition theorem.



Step1: Consider 12V only and short 12V and 8V source.



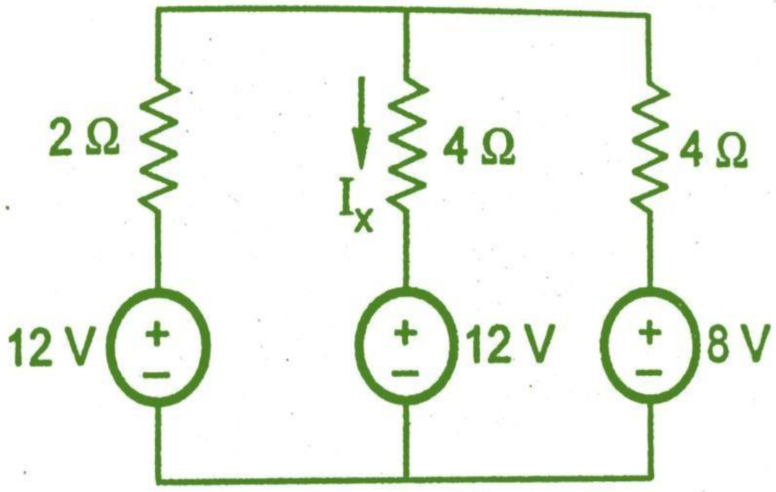


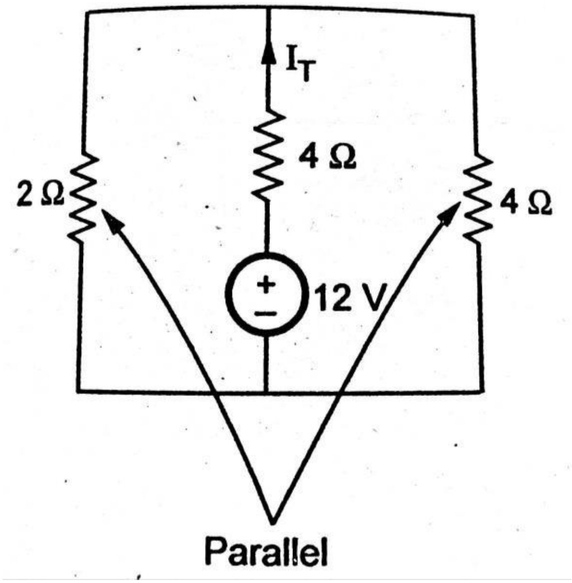
$I_T = \frac{12}{2+2}$ = 3A

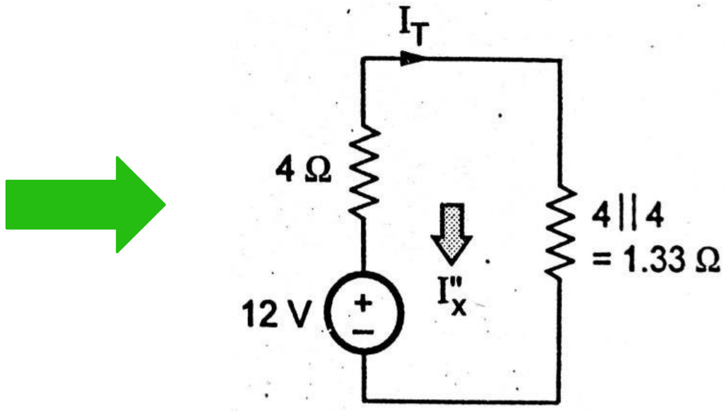
Applying Current Division Rule

$I'_x = I_T$ X $\frac{4}{4+4}$ = 3/2 = 1.5 A

Step2: Consider 12V only and short 12V and 8V source.



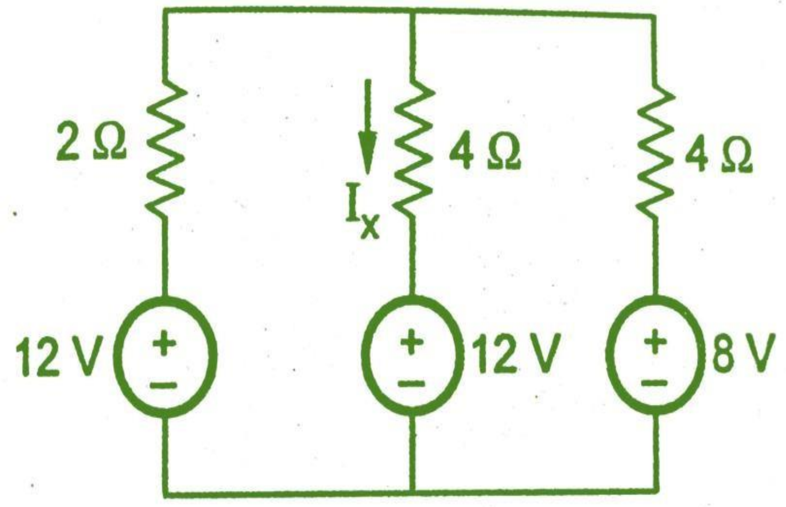


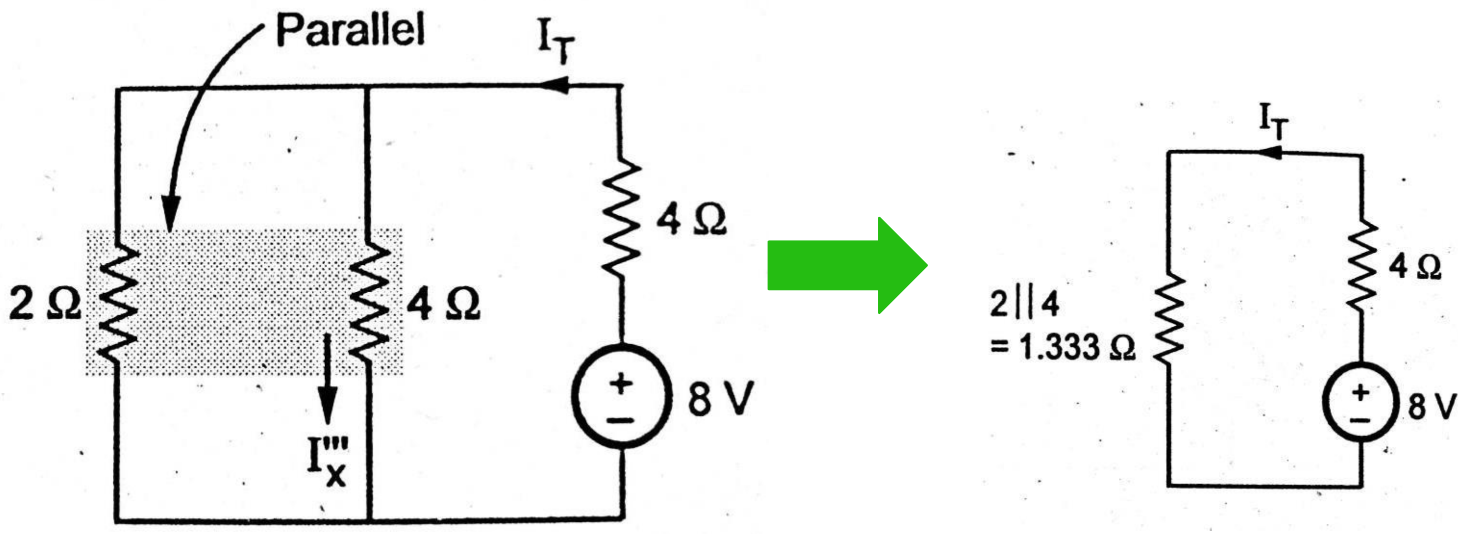


$$I_T = \frac{12}{4+1.33} = 2.25\text{A}$$

$$I''_x = -I_T = -2.25\text{ A}$$

Step3: Consider 8V only and short 12V and 12V source.





= 1.5A

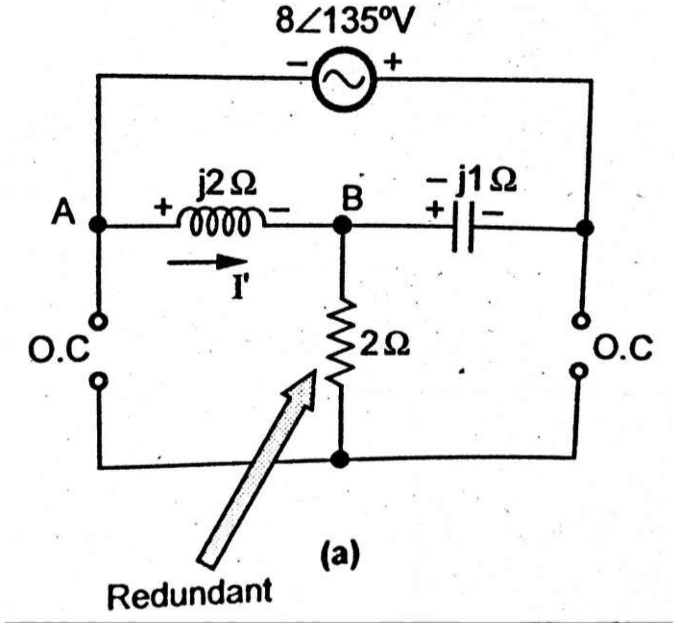
Applying Current Division Rule

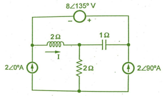
$$I'''_x = I_T \times \frac{2}{2+4} = \frac{1.5 \, X \, 2}{2+4} = 0.5 \text{ A}$$

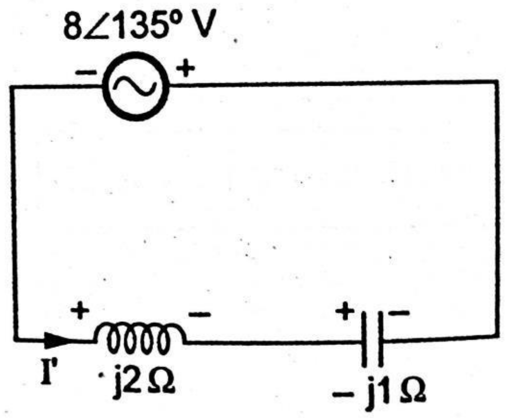
$$I_x = I'_x + I''_x + I'''_x = 1.5 - 2.25 + 0.5 = -0.25 \text{ A}$$

Problem5: Using superposition theorem, obtain the response 'I' for the network shown.

Step1: Consider 8∠135 only and open 2∠0 and 2∠90 source.





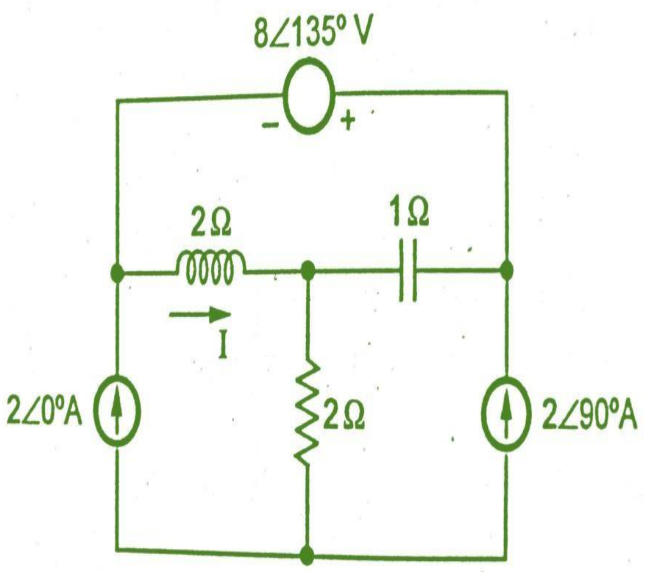


Applying KVL in the direction of I'

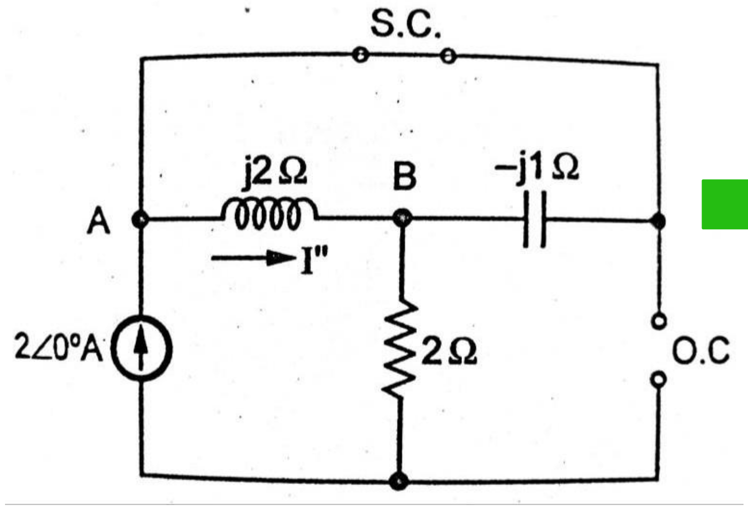
$-(2j)\, I' - (-j)\, I' - 8\angle 135 = 0$

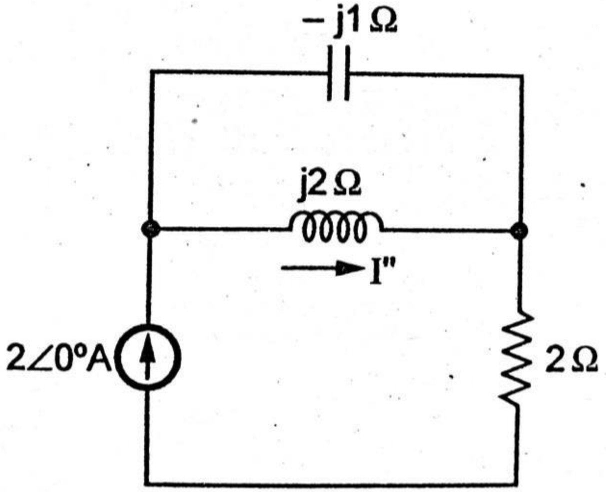
$-j\, I' = 8\angle 135$

$I' = 8\angle -135$ A



Step2: Consider 2∠0 only and open 2∠90 and short 8 ∠ 135 source.



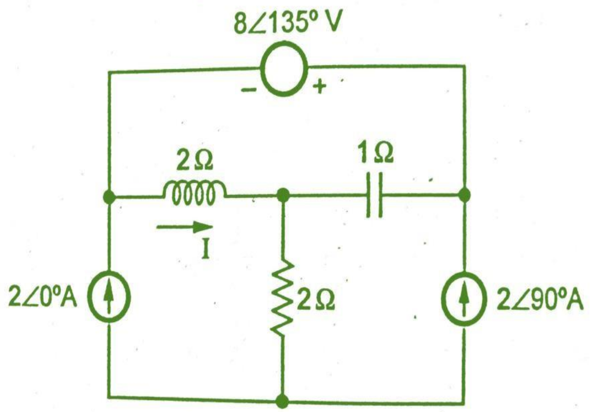


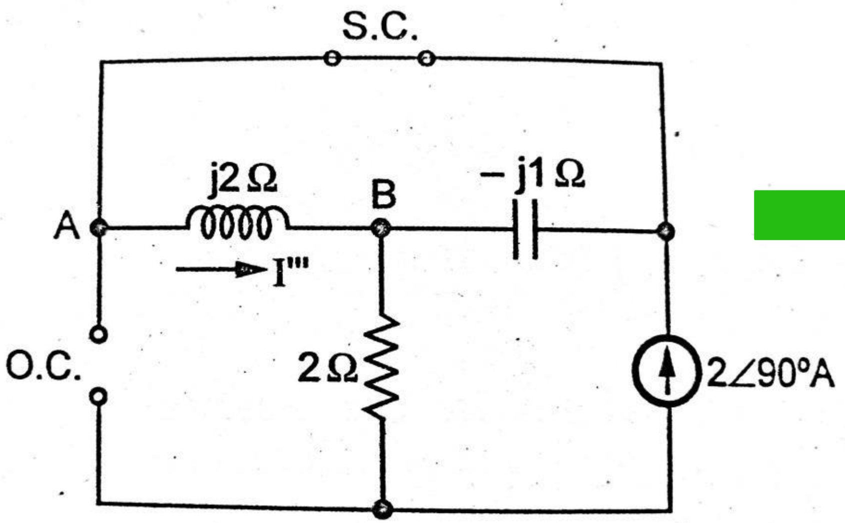
Using Current divider rule,

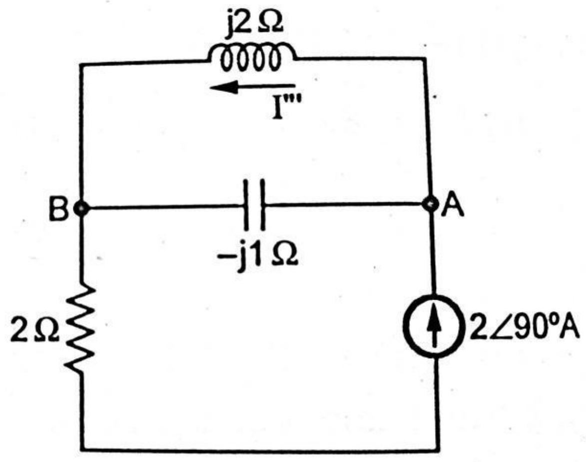
$$I'' = 2\angle 0 \text{ X } \frac{(-j1)}{-j1+j2}$$

I'' = -2 A

Step3: Consider 2∠90 only and open 2∠0 and short 8 ∠ 135 source.







Using Current divider rule,

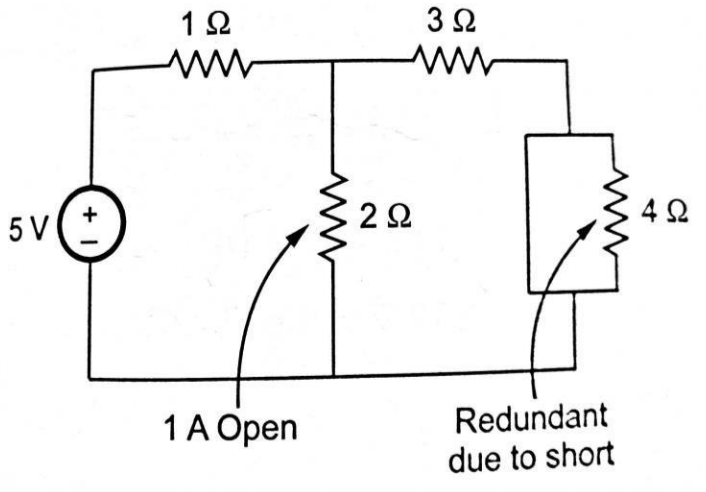
$$I''' = 2\angle 90 \text{ X } \frac{(-j1)}{-j1+j2}$$

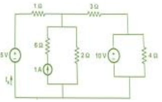
I''' = 2 ∠ -90 A

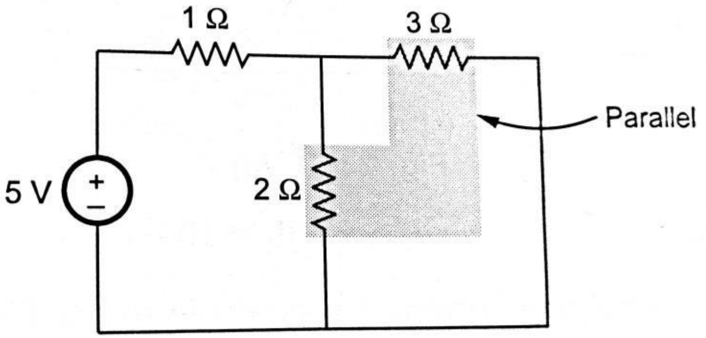
I = I' + I'' + I''' = 8 ∠-135 - 2 + 2 ∠ -90 = 10.82 ∠ -135 A

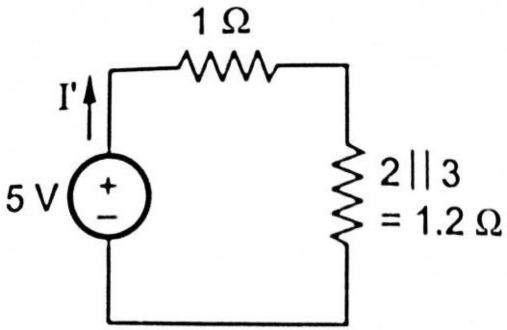
Problem6: **Find 'I_x' using superposition theorem.**

Step1: **Consider 5V only and open 1A and short 10V source.**





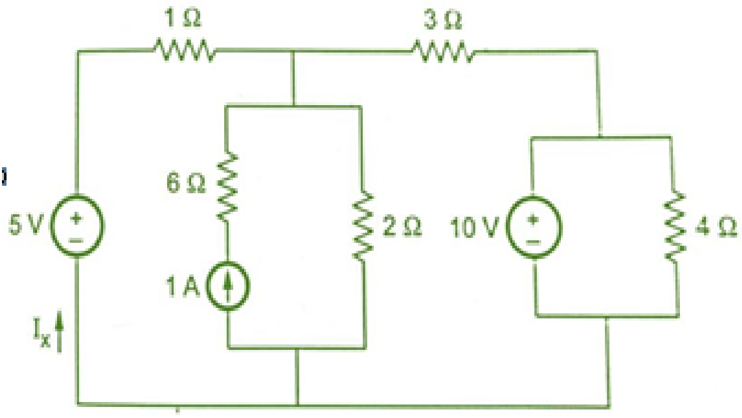




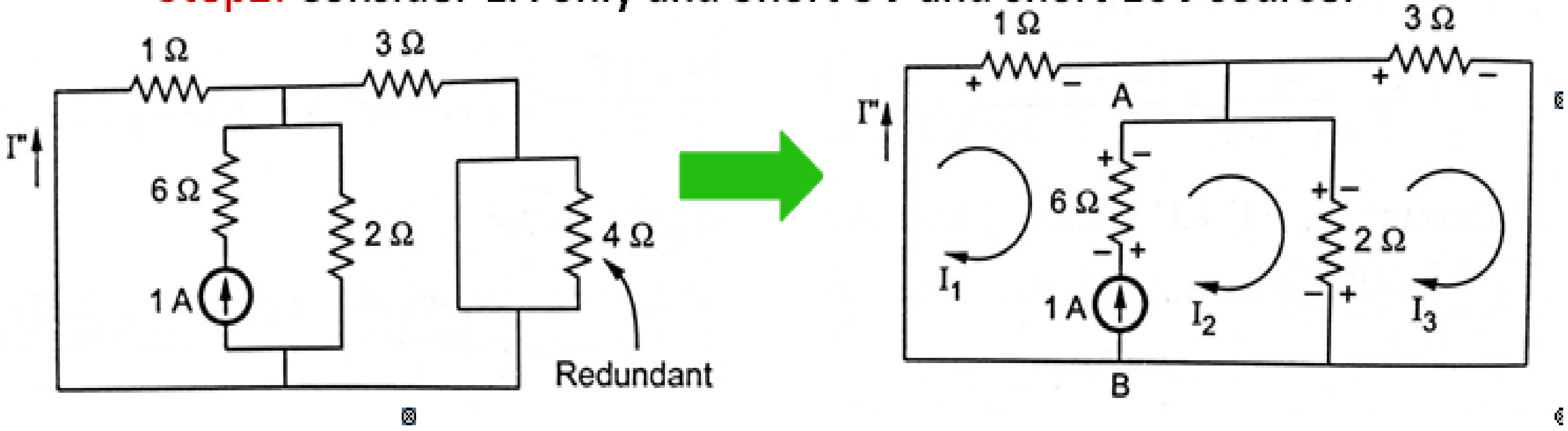
Using Ohm's law,

$$I'_x = \frac{5}{1+1.2}$$

$$I'_x = 2.2727 \text{ A}$$



Step2: Consider 1A only and Short 5V and short 10V source.



I_2 - I_1 = 1 ----------------------------(1)

Applying KVL

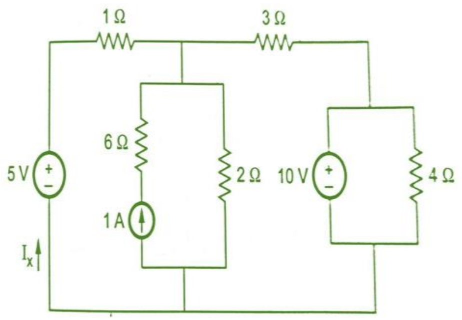
-I_1 - 2I_2 + 2I_3 = 0 ----------------(2)

-3I_3 - 2I_3 + 2I_2 = 0

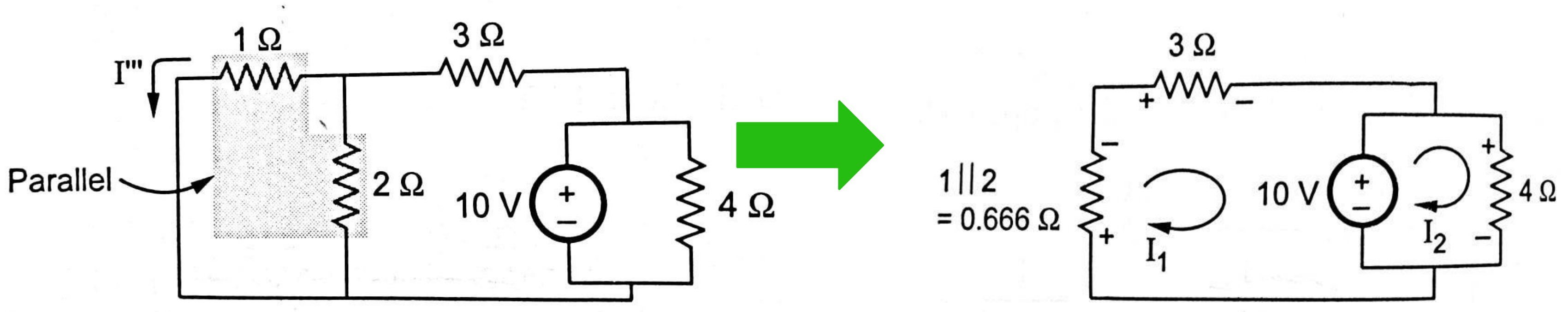
$2I_2$ - $5I_3$ = 0 ----------------------(3)

I_1= -0.5454A, I_2= 0.4545 A, I_3= 0.1818 A

I''_x = I_1 = -0.5454A



Step3: Consider 10V only and Short 5V and open 1A source.



Applying KVL

$-0.66I_1 - 3I_1 - 10 = 0$

$-3.66\, I_1 = 10$

$I_1 = \dfrac{10}{-3.66} = -2.7322$ A

$I = -I_1 = 2.7322$ A

Using Current division rule,

$I''' = I \times \dfrac{2}{1+2} = 1.8214$ A

$I'''_x = -I''' = -1.8214$ A

$I_x = I'_x + I''_x + I'''_x = 2.2727 - 0.5454 - 1.8215 = -0.09$ A

Millman's Theorem

If n voltage sources V_1, V_2, ……. V_n having internal impedances (or series impedances) Z_1, Z_2, …… Z_n respectively, are in parallel, then these sources may be replaced by a single voltage source of Voltage V_M having a series impedance Z_M where V_M and Z_M are given by

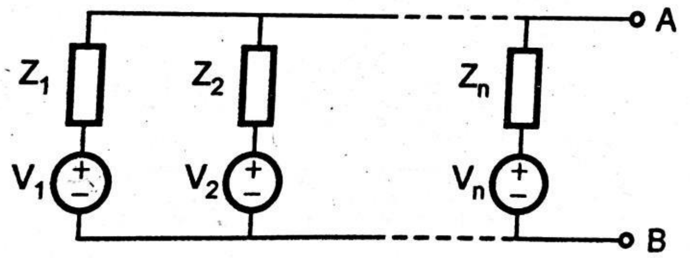
$$V_M = \frac{V_1Y_1+V_2Y_2+ \ldots\ldots+V_nY_n}{Y_1+Y_2+ \ldots\ldots+Y_n} = \frac{\sum_{k=1}^{n} V_kY_k}{\sum_{k=1}^{n} Y_k}$$

And

$$Z_M = \frac{1}{Y_1+Y_2+ \ldots\ldots+Y_n} = \frac{1}{\sum_{k=1}^{n} Y_k}$$

Proof of Millman's Theorem

Consider n voltage sources in parallel as shown in the figure.



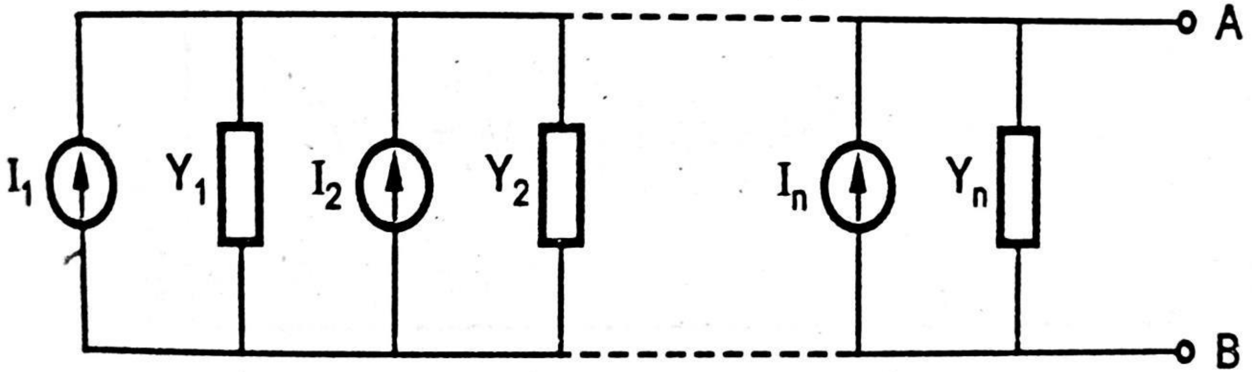
Let us convert each voltage source into an equivalent current source for source 1.

$$I_1 = \frac{V_1}{Z_1} = V_1\ Y_1 \text{ as } Y_1 = \frac{1}{Z_1}$$

Similarly for the remaining sources, we can write

$$I_2 = V_2 Y_2\ ,\ I_3 = V_3 Y_3\ ,\ ....\ I_n = V_n Y_n$$

Where $Y_1, , ... Y_n$ are the admittances to be connected in parallel. Hence circuit reduces to

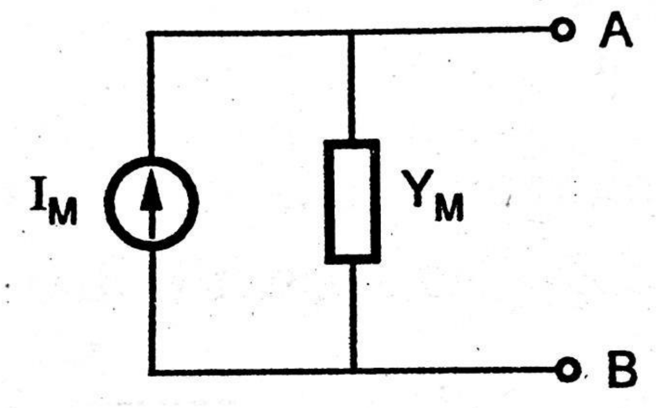


Hence the effective current source across the terminal AB is,

$I_M = I_1 + I_2 + \ldots + I_n$ --------------------------(1)

$Y_M = Y_1 + Y_2 + \ldots + Y_n$ --------------------------(2)

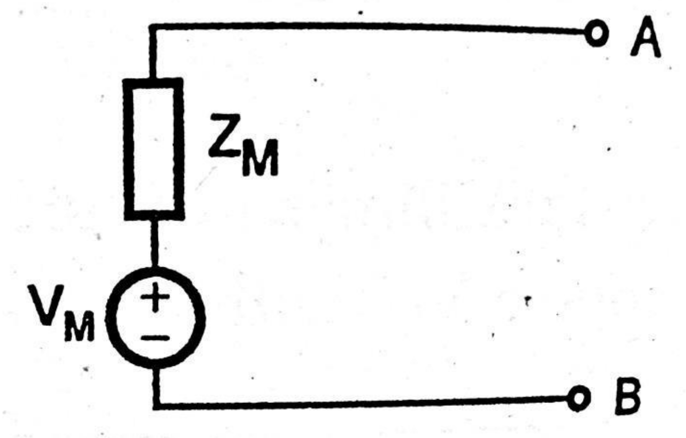
This is because admittance in parallel get added to each other. Hence the circuit reduces to,



Converting this equivalent current source into the voltage source we get,

$V_M = \frac{I_M}{Y_M}$ and as $Z_M = \frac{1}{Y_M}$

$V_M = I_M Z_M$



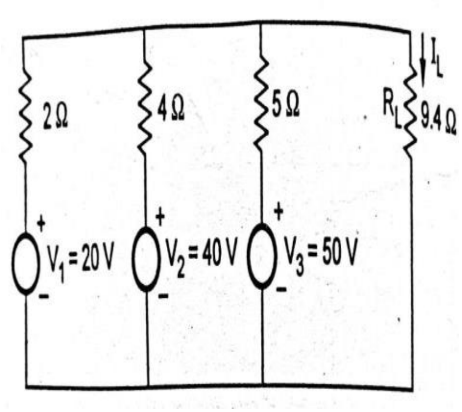
Substituting I_M and Y_M from (1) and (2),

$$V_M = (I_1 + I_2 + \ldots + I_n)\frac{1}{(Y_1 + Y_2 + \ldots + Y_n)}$$

But $I_1 = \frac{V_1}{Z_1} = V_1 Y_1$, $I_2 = V_2 Y_2$, ... $I_n = V_n Y_n$

$$V_M = \frac{V_1Y_1 + V_2Y_2 + \ldots\ldots + V_nY_n}{Y_1 + Y_2 + \ldots\ldots + Y_n}, \quad Z_M = \frac{1}{Y_1 + Y_2 + \ldots\ldots + Y_n}$$

Problem1: Using Millman's Theorem, find I_L through R_L for the network shown in figure .



For the given network, we can write,

$V_1 = 20$ V, $\quad Z_1 = 2\ \Omega$, $\quad$ Hence $Y_1 = \frac{1}{Z_1} = \frac{1}{2}$ ℧

$V_2 = 40$ V, $\quad Z_2 = 4\ \Omega$, $\quad$ Hence $Y_2 = \frac{1}{Z_2} = \frac{1}{4}$ ℧

$V_3 = 50$ V, $\quad Z_3 = 5\ \Omega$, $\quad$ Hence $Y_3 = \frac{1}{Z_3} = \frac{1}{5}$ ℧

According to Millman's Theorem,

$$Z_M = \frac{1}{Y_1 + Y_2 + Y_3} = \frac{1}{\frac{1}{2}+\frac{1}{4}+\frac{1}{5}} = 1.0526\ \Omega$$

$$V_M = \frac{V_1Y_1 + V_2Y_2 + V_3Y_3}{Y_1 + Y_2 + Y_3} = \frac{20(0.5)+40\ (0.25)50\ (0.2)}{0.95}$$

$$= 31.5789\ \text{V}$$

$$I_L = \frac{V_M}{Z_M + R_L} = 3.0211\ \text{A}$$

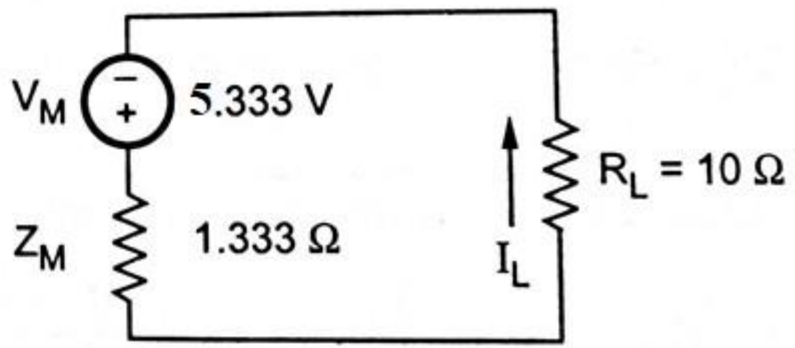
Problem2: Using Millman's Theorem, find current through R_L for the network shown in figure .

$$Z_1 = 4\ \Omega = Z_2 = Z_3$$

$$Y_1 = Y_2 = Y_3 = \frac{1}{4}\text{ mho} = 0.25\text{ mho}$$

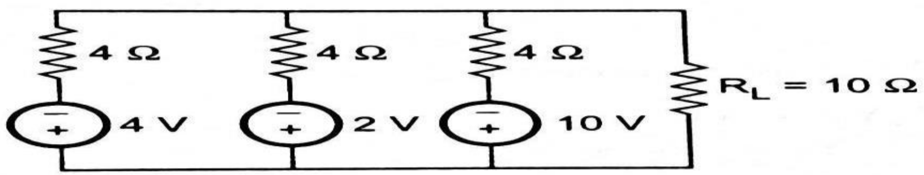
$$Z_M = \frac{1}{Y_1 + Y_2 + Y_3} = \frac{1}{\frac{3}{4}} = 1.333\ \Omega$$

$$I_L = \frac{V_M}{R_L + Z_M} = \frac{5.333}{10 + 1.333} = \mathbf{0.47A}$$

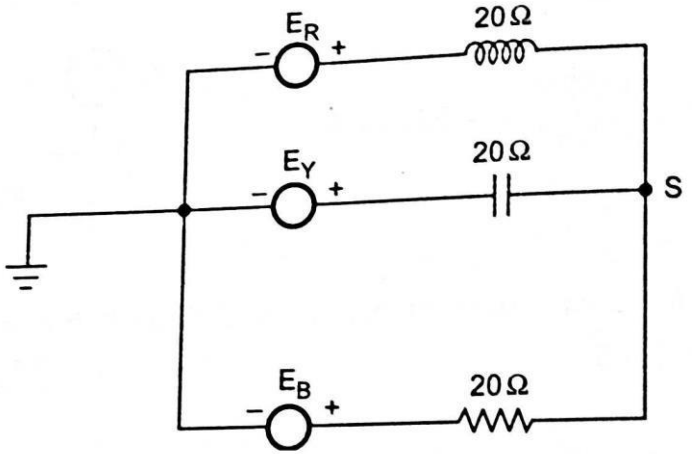


Now let V_1 =10V, V_2 = 2V, V_3= 4V

$$V_M = \frac{V_1 Y_1 + V_2 Y_2 + V_3 Y_3}{Y_1 + Y_2 + Y_3} = \frac{10\times\frac{1}{4} + 2\times\frac{1}{4} + 4\times\frac{1}{4}}{\frac{3}{4}} = 5.333\text{ V}$$



Problem3: Using Millman's Theorem, determine the voltage V_S of the network shown in figure . Given E_R= 230∠0, E_Y = 230∠-120, E_B = 230∠+120



$$V_M = \frac{V_1Y_1 + V_2Y_2 + V_3Y_3}{Y_1 + Y_2 + Y_3}$$

$V_1 = E_R = 230\angle 0°\,V,\ V_2 = E_Y = 230\angle -120°\,V,\ V_3 = E_B = 230\angle +120°\,V$

$$Z_1 = +j20\ \Omega,\ \text{i.e. } Y_1 = \frac{1}{Z_1} = \frac{1}{j20} = -j\,0.05\ \text{mho}$$

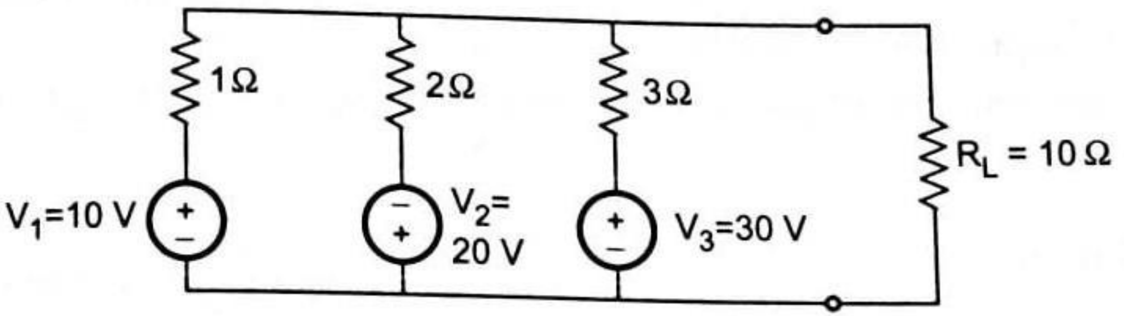
$$Z_2 = -j20\ \Omega,\ \text{i.e. } Y_2 = \frac{1}{Z_2} = \frac{1}{(-j20)} = +j\,0.05\ \text{mho}$$

$$Z_3 = 20\ \Omega,\quad \text{i.e. } Y_3 = \frac{1}{Z_3} = \frac{1}{20} = 0.05\ \text{mho}$$

$$V_M = \frac{230\angle 0° \times (-j\,0.05) + (230\angle -120°)(+j\,0.05) + (230\angle +120°)(0.05)}{-j\,0.05 + j\,0.05 + 0.05}$$

$$V_S = V_M = 168.372\angle -60°\,V$$

Problem4: Using Millman's Theorem, determine the current through R_L of the network shown in figure .



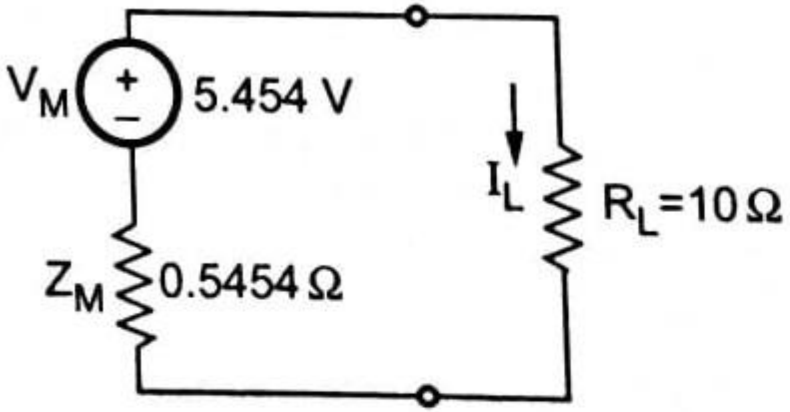
$Z_1 = 1\ \Omega,\ Z_2 = 2\ \Omega$ and $Z_3 = 3\ \Omega$

$Y_1 = 1$ mho, $Y_2 = \frac{1}{2}$ mho and $Y_3 = \frac{1}{3}$ mho

$$Z_M = \frac{1}{Y_1 + Y_2 + Y_3} = \frac{1}{1 + \frac{1}{2} + \frac{1}{3}} = 0.5454\ \Omega$$

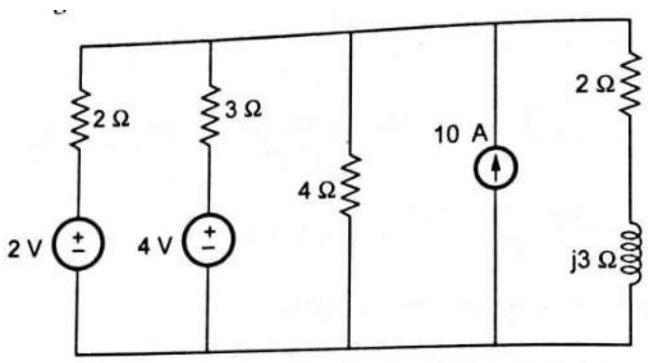
As the sign of V_2 is opposite to V_1 and V_3,

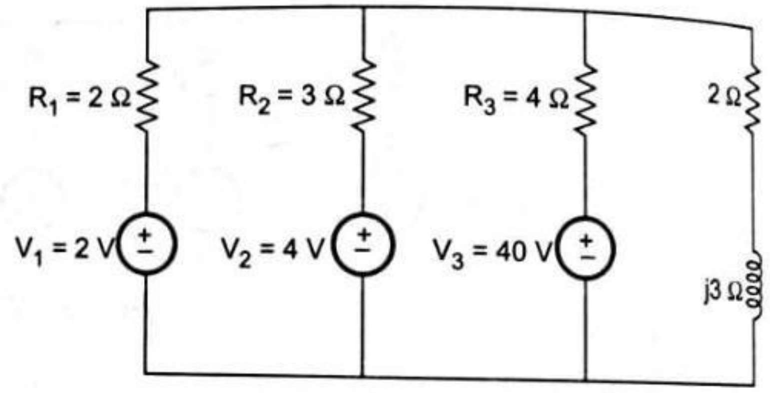
$$V_M = \frac{V_1Y_1 - V_2Y_2 + V_3Y_3}{Y_1 + Y_2 + Y_3} = 5.454\ \text{V}$$



$$I_L = \frac{V_M}{Z_M + R_L} = 0.5172\ \text{A}$$

Problem5: Using Millman's Theorem, find the current through (2+3j) of the network shown in figure .





orem,

For the given network, we can write,

$V_1 = 2$ V, $Z_1 = 2\ \Omega$, Hence $Y_1 = \frac{1}{Z_1} = \frac{1}{2}$ ℧

$V_2 = 4$ V, $Z_2 = 3\ \Omega$, Hence $Y_2 = \frac{1}{Z_2} = \frac{1}{3}$ ℧

$V_3 = 40$ V, $Z_3 = 4\ \Omega$, Hence $Y_3 = \frac{1}{Z_3} = \frac{1}{4}$ ℧

$$Z_M = \frac{1}{Y_1 + Y_2 + Y_3} = \frac{1}{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}} = 0.923\ \Omega$$

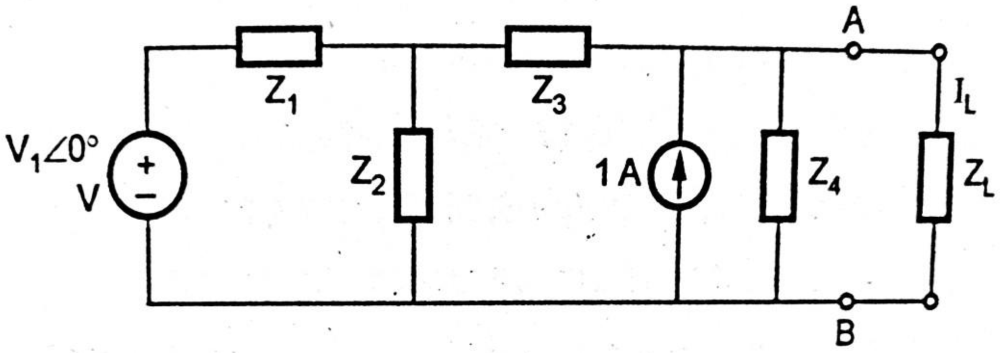
$$V_M = \frac{V_1Y_1 + V_2Y_2 + V_3Y_3}{Y_1 + Y_2 + Y_3} = \frac{2(0.5)+4\,(0.33)+40\,(0.25)}{1.08}$$

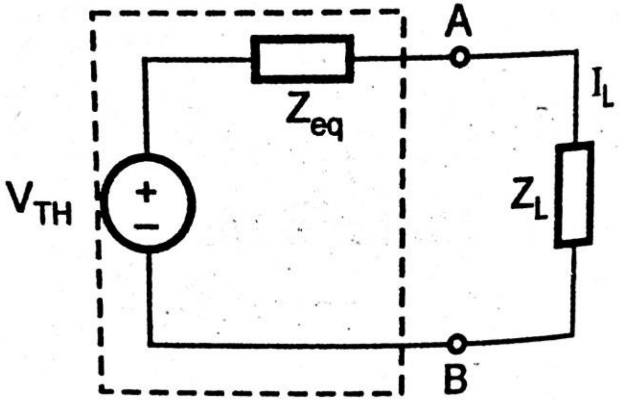
$$= 11.38\ \text{V}$$

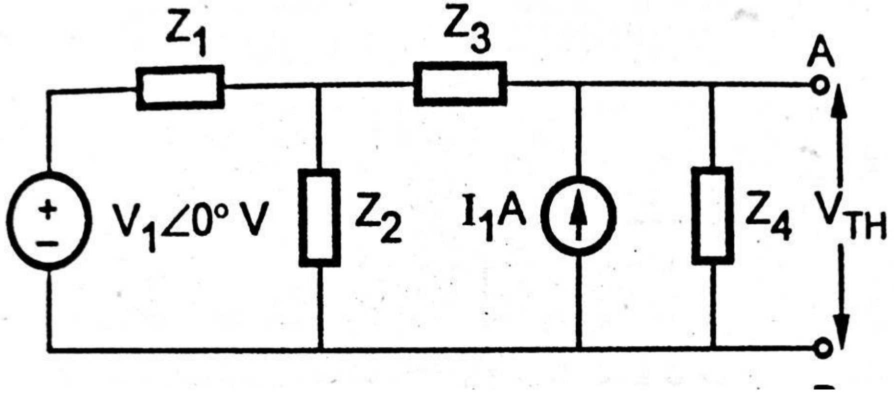
$$I_L = \frac{V_M}{Z_M + R_L} = 2.7\ \angle{-45.74}\ \text{A}$$

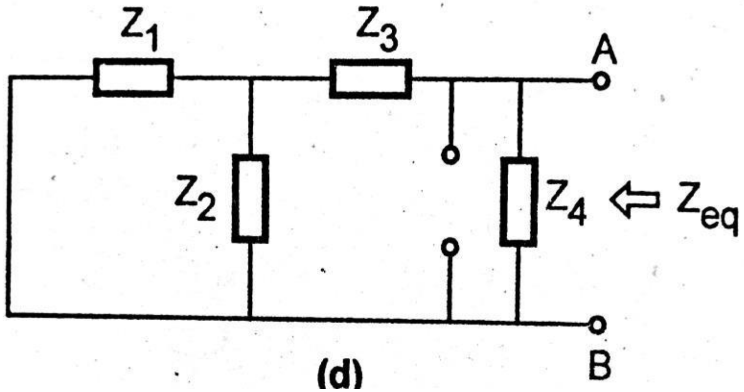
Thevenin's Theorem

"Any linear circuit containing several voltages and resistances can be replaced by just one single voltage in series with a single resistance connected across the load".









(d)

Steps to apply Thevenin's Theorem

- ✓ **Remove the branch impedance, through which current is required to be calculated.**

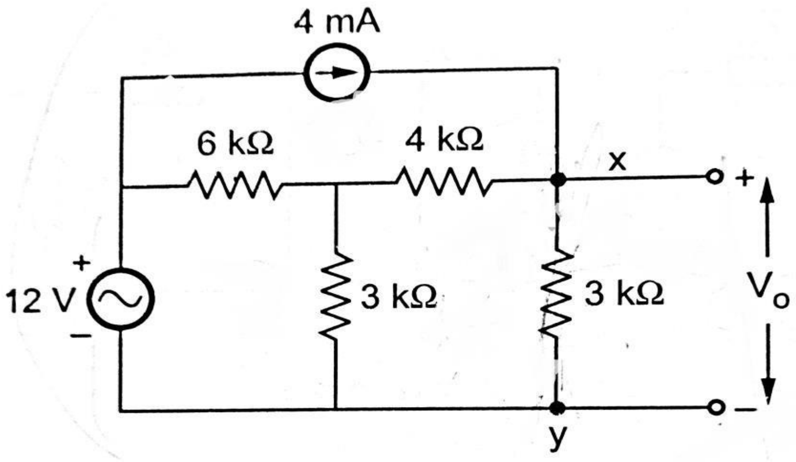
- ✓ **Calculate the voltage across these open circuited terminals, by using any of the network simplification techniques. This voltage is Thevenin's equivalent voltage V_{TH}.**

- ✓ **Calculate the equivalent impedance Z_{eq}, as viewed through the two terminals of the branch from which current is to be calculated by removing that branch impedance and replacing all the independent sources by their internal imprdances.**

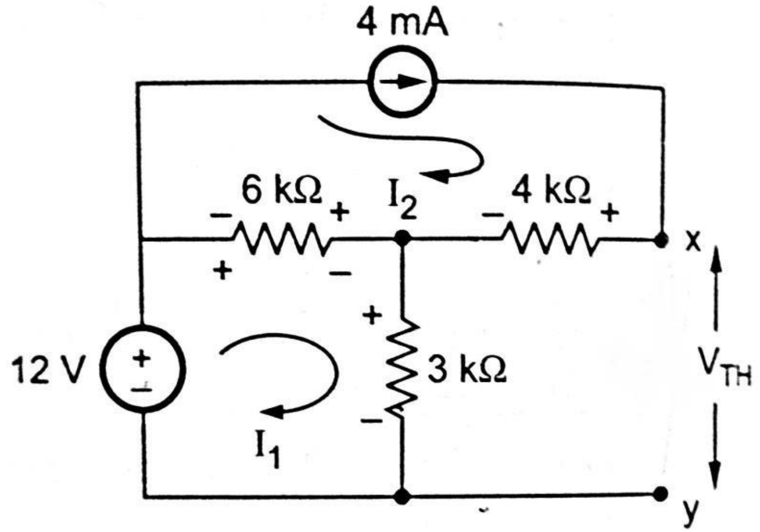
- ✓ **Draw the Thevenin's equivalent showing the voltage source V_{TH}, with the impedance Z_{eq} in series with it, across the terminals of the branch through which the current is to be calculated. Reconnect the branch impedance now. Let it be Z_L. The required current through the branch is given by,**

$$I = \frac{V_{TH}}{Z_L + Z_{eq}}$$

Problem1: Obtain the Thevenin's equivalent of network shown in fig between terminals x and y.Also find V_0.



Remove 3kΩ resistance connected between x and y.



Find V :

From the current source branch,

I_2 = 4mA

I_1= 4mA

Applying KVL,

-6k I_1 + 6k I_2 - 3kI_1+12=0

Using I_2 = 4mA

The voltage drop across 3kΩ

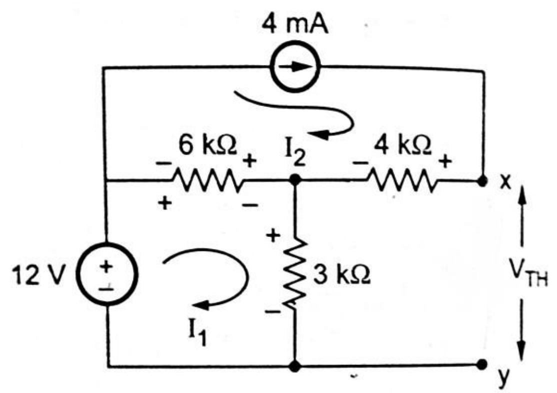
is = 3k X 4m = 12V

The voltage drop across 4kΩ

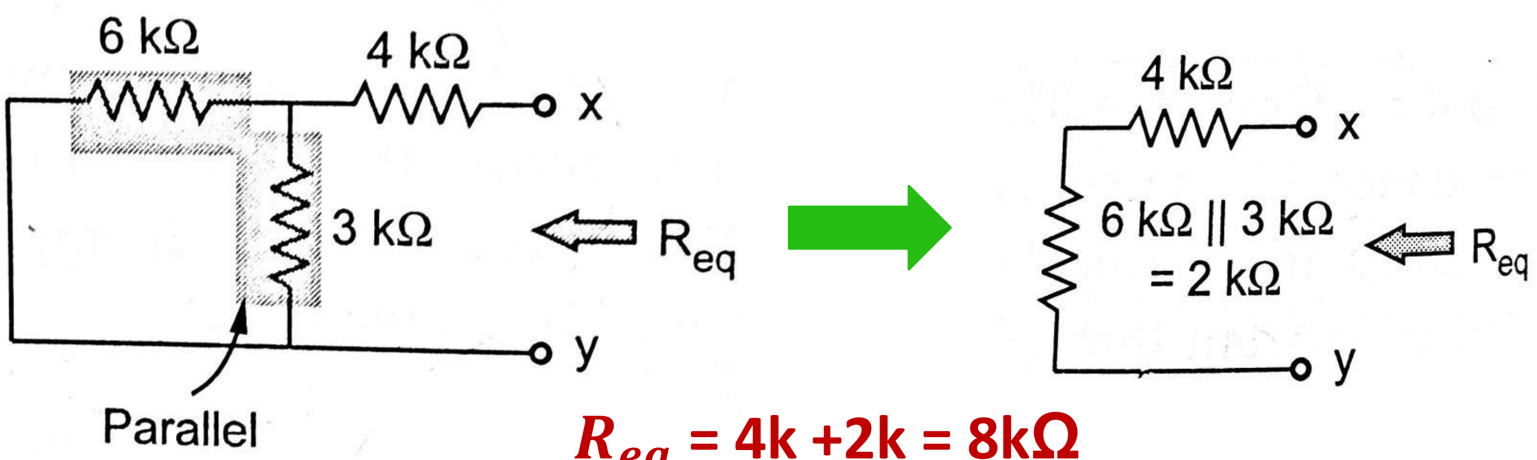
is = 4k X 4m = 16V

Trace the path x-y

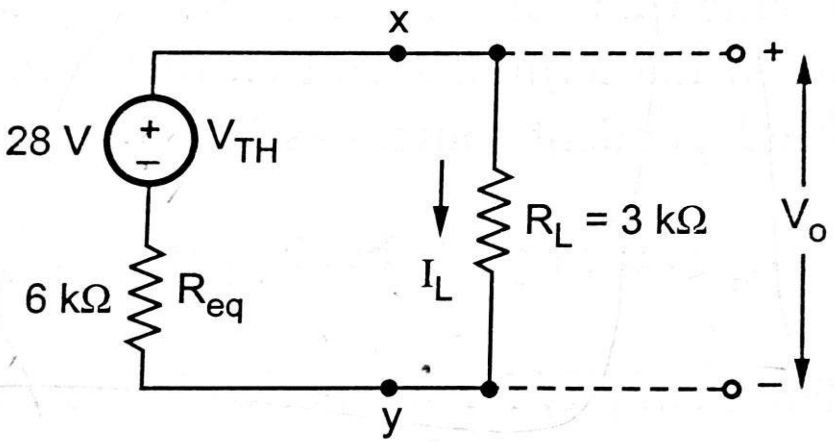
V_{TH}= V_0 = 28V



To find R_{eq} short the voltage source and open current source



R_{eq} = 4k +2k = 8kΩ



The Thevenin's equivalent circuit is shown in fig.

$$I_L = \frac{V_{TH}}{Z_L + Z_{eq}}$$

= 3.111 mA

V_0 = 3.111m X 3k

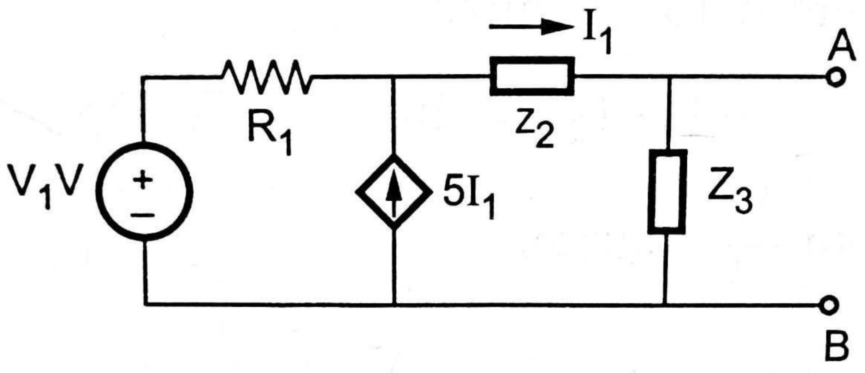
= 9.333V

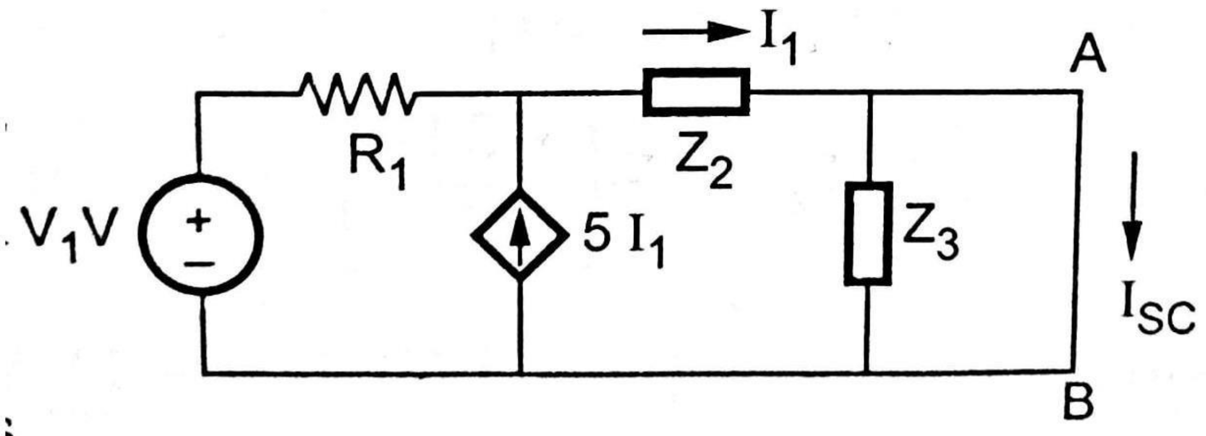
Method of calculating Z_{eq} for network with dependent sources

Z_{eq} can be calculated as

$$Z_{eq} = \frac{V_{OC}}{I_{SC}}$$

$$V_{OC} = V_{TH}$$

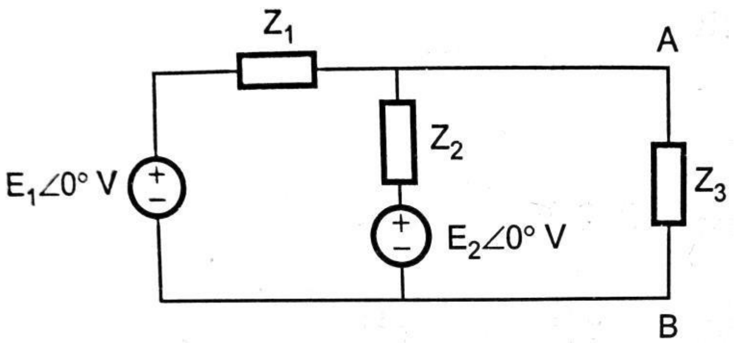




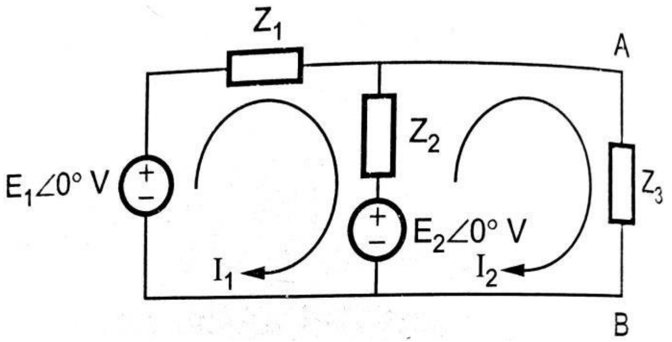
While calculating I_{SC}, all the independent as well as dependent sources must be kept as it is.
None of the sources are replaced by open or short circuit.

Proof of Thevenin's Theorem

Consider the network shown.



Let us find the current through Z_3 by mesh analysis first. Assume the currents as shown.



Applying KVL to the loop,

$-Z_1 I_1 - Z_2 I_1 + Z_2 I_2 - E_2 + E_1 = 0$

$(-Z_1 - Z_2) I_1 + Z_2 I_2 = E_2 - E_1$ ------------(1)

$-Z_3 I_2 - Z_2 I_2 + Z_2 I_1 + E_2 = 0$

$Z_2 I_1 + (-Z_3 - Z_2) I_2 = -E_2$ -----------------(2)

$$D = \begin{vmatrix} -Z_1 - Z_2 & Z_2 \\ Z_2 & -Z_3 - Z_2 \end{vmatrix} = Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1$$

Now current through Z_3 is I_2, hence calculating D_2,

$$D_2 = \begin{vmatrix} -Z_1 - Z_2 & E_2 - E_1 \\ Z_2 & -E_2 \end{vmatrix} = Z_1 E_2 + Z_2 E_1$$

$$I_2 = \frac{D_2}{D}$$

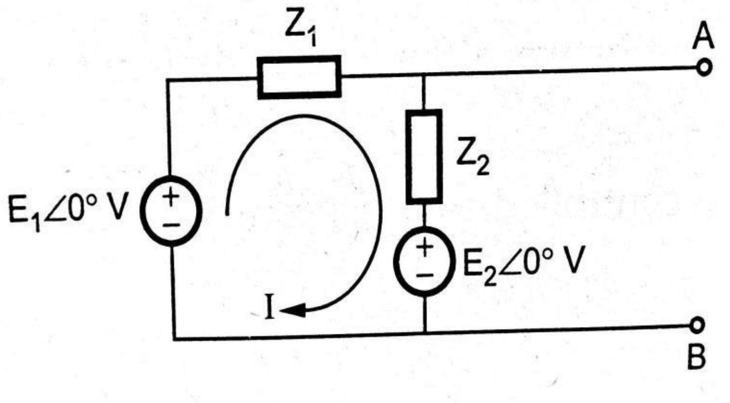
$$= \frac{Z_1 E_2 + Z_2 E_1}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}$$

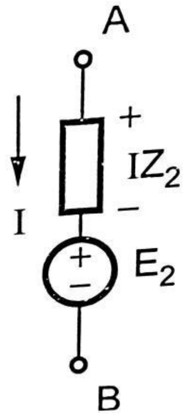
Required current

Let us use Thevenin's Theorem.

Step1: **Remove the impedance Z_3, through which current is to be calculated.**

Step2: **Obtain the open circuit voltage V_{AB} = V_{TH}**





-IZ_1 - IZ_2 - E_2 + E_1 = 0

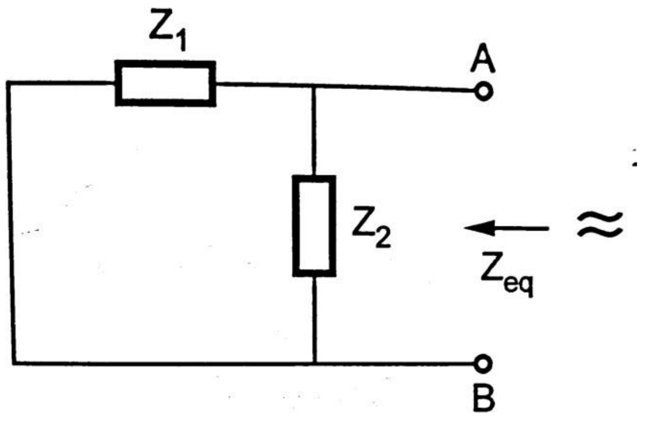
$$I = \frac{E_1 - E_2}{Z_1 + Z_2}$$

$$V_{AB} = IZ_2 + E_2$$

$$= \frac{(E_1 - E_2)Z_2}{Z_1 + Z_2} + E_2 = V_{TH}$$ **with A positive**

Step3: Obtain Z_{eq} as viewed through terminals A-B, with both the sources replaced by short circuit.

$$Z_{eq} = (Z_1 \,||\, Z_2) = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

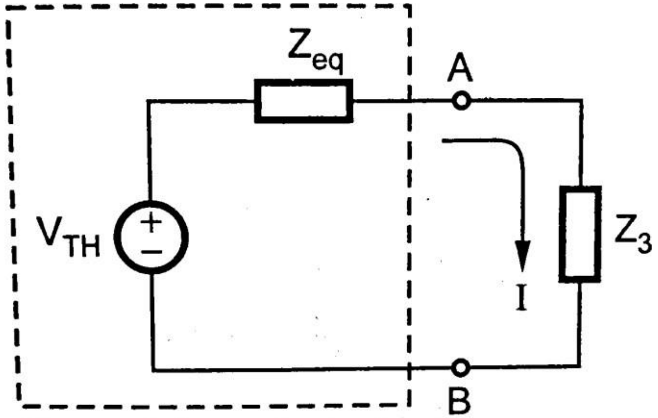


Step4: Thevenin's equivalent across the terminals AB is as shown. Hence the required current I is,

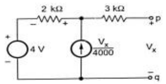
$$I = \frac{V_{TH}}{Z_3 + Z_{eq}}$$

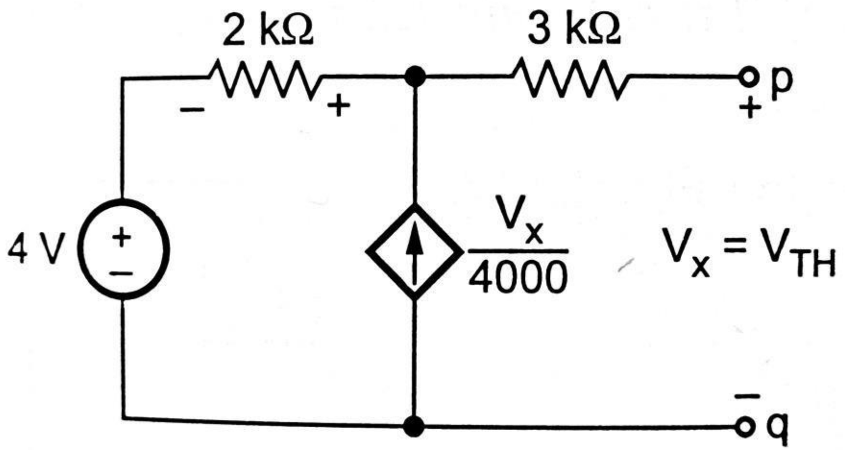
$$= \frac{\frac{(E_1 - E_2) Z_2}{Z_1 + Z_2} + E_2}{Z_3 + \frac{Z_1 Z_2}{Z_1 + Z_2}}$$

$$I = \frac{Z_1 E_2 + Z_2 E_1}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}$$



Problem2: Obtain the Thevenin's equivalent of network shown in fig between terminals p and q.

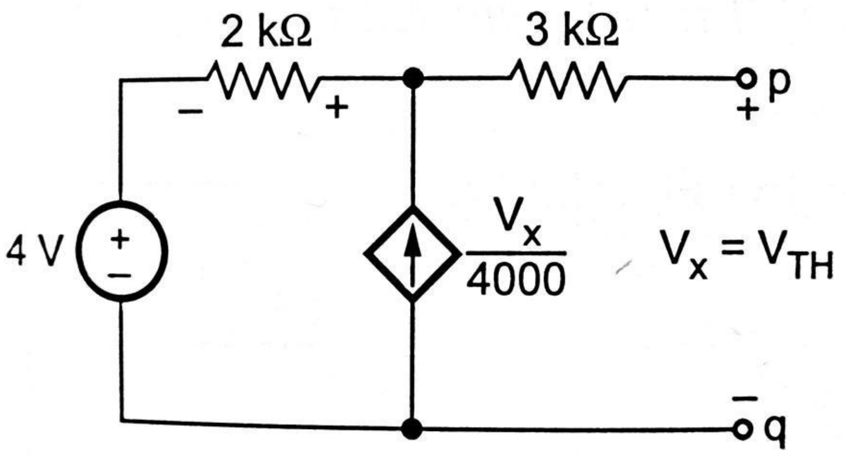


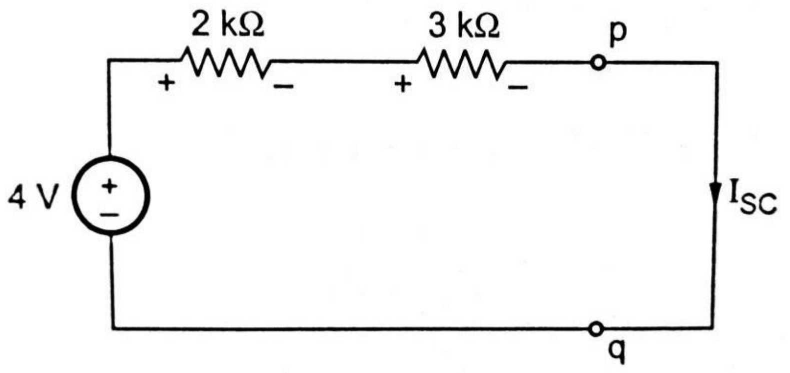


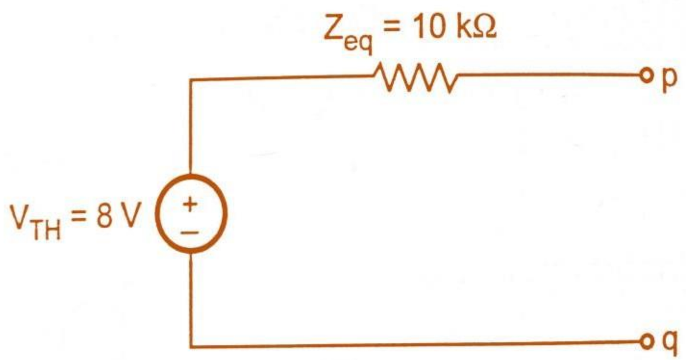
$$V_x = (2 \times 10^3)\,\frac{Vx}{4000} - 4$$

$$V_x - \frac{Vx}{2} = 4$$

$$V_x = 8\text{ V} = V_{TH}$$







Applying KVL to the loop,

-(2 x 10³) I_{SC} - (3 x 10³) I_{SC} + 4 = 0

- (5 x 10³) I_{SC} = -4

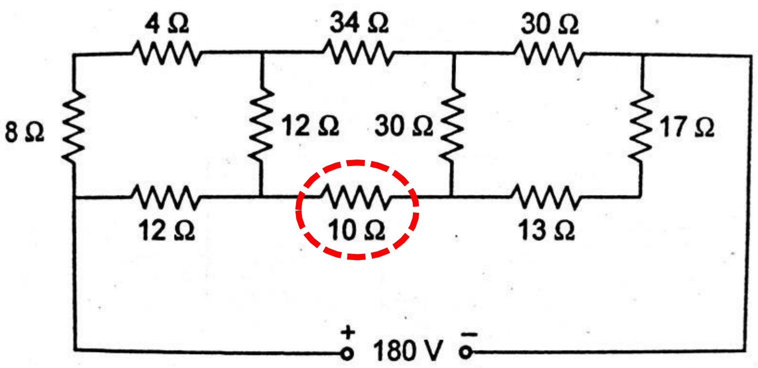
I_{SC} = 0.8 mA

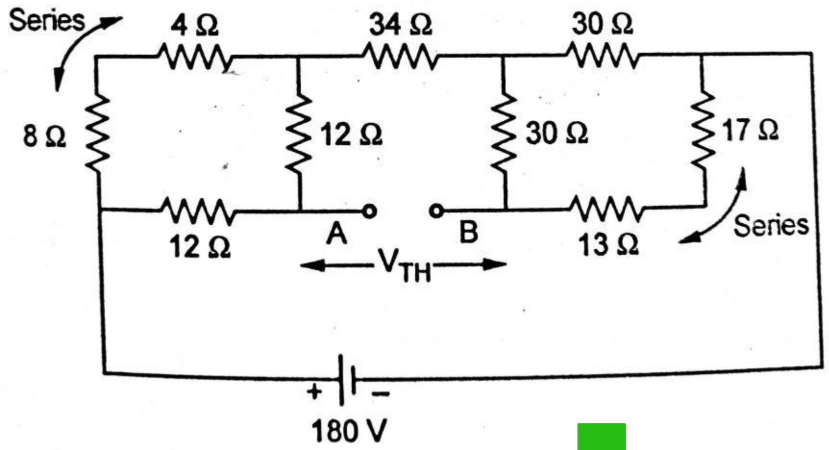
$$Z_{eq} = \frac{V_{TH}}{I_{SC}}$$

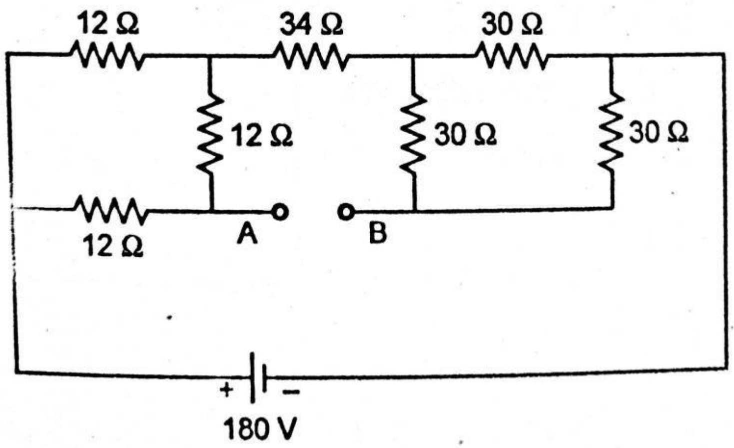
$$Z_{eq} = \frac{8}{0.8X10^{-3}}$$

Z_{eq} = 10kΩ

Problem3: Find the current in the 10Ω resistor in the network shown by using Thevenin's Theorem.





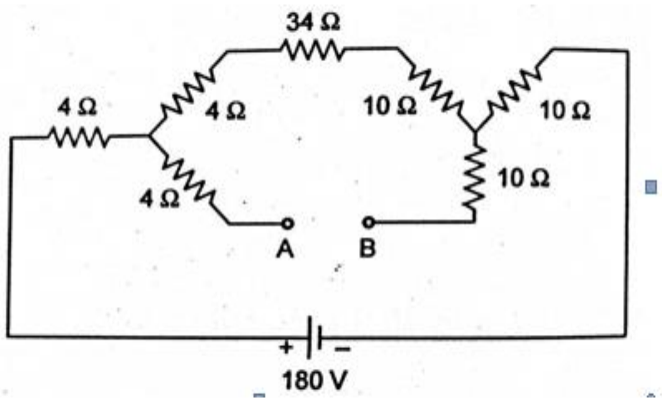


Convert 12Ω Delta to star,

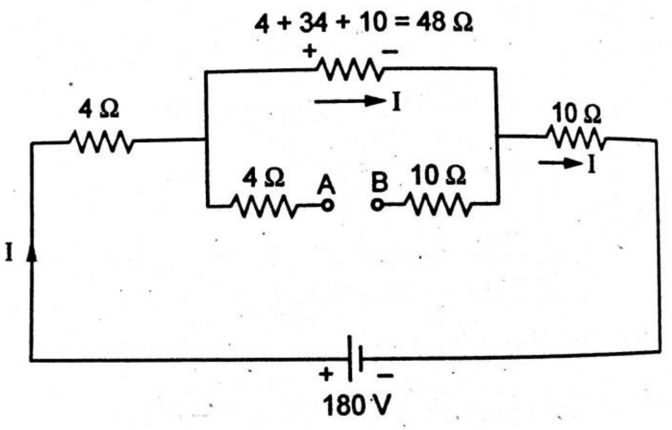
$$\frac{12X12}{12+12+12} = 4\Omega$$

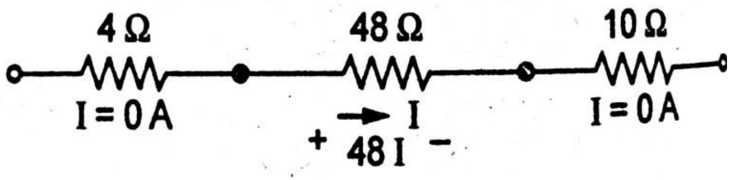
Convert 30Ω Delta to star,

$$\frac{30X30}{30+30+30} = 10\Omega$$

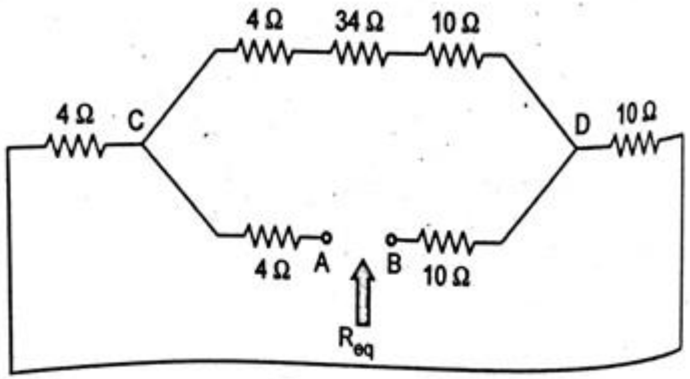


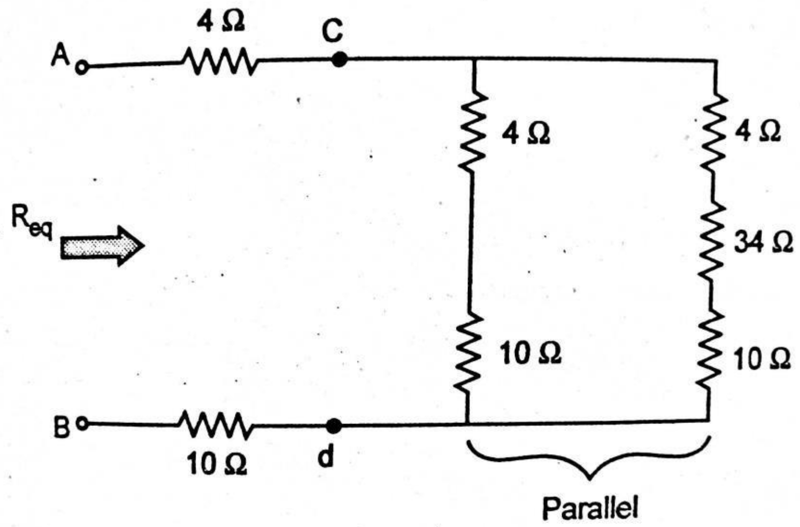
$$I = \frac{180}{4+48+10} = 2.9032A$$





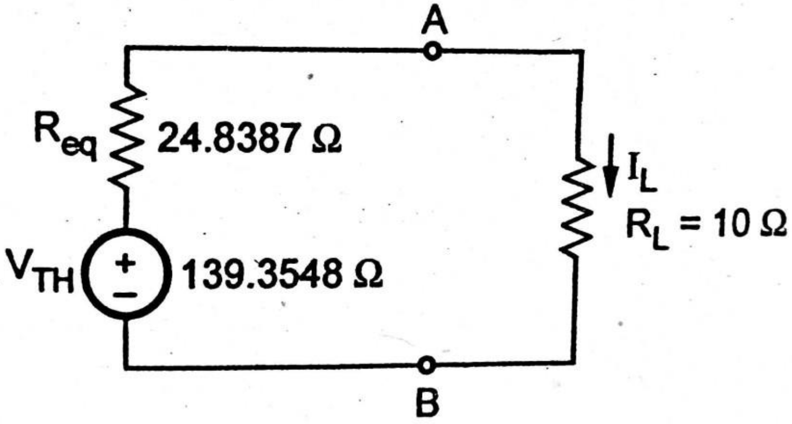
$$V_{TH} = V_{AB} = 48\ I = 139.35\ V$$





R_{eq} = 4+(14||48)+10

= 24.8387Ω

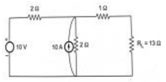


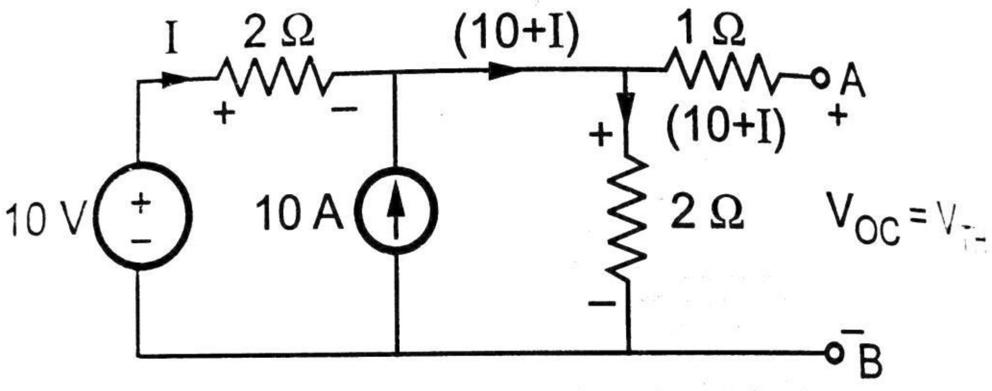
$$I = \frac{V_{TH}}{R_L + R_{eq}}$$

$$= \frac{139.3548}{10+24.8387}$$

= 4A

Problem4: Find the current through load resistance using Thevenin's Theorem.

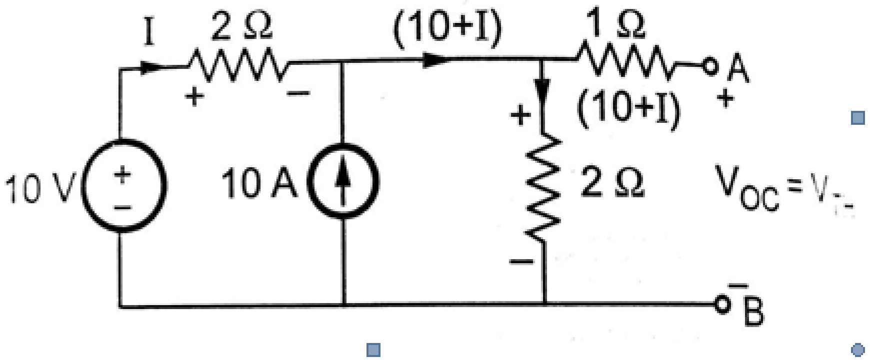


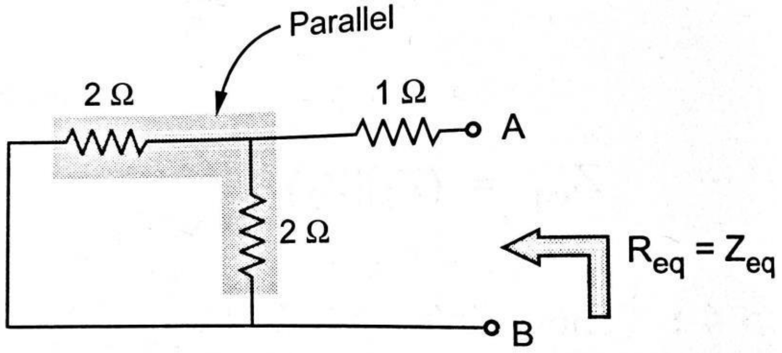


Applying KVL to outer closed path excluding current source, we get

-2I – 2(10+I) +10 = 0
-2I – 20 - 2I +10 = 0
-4I = 10
I = -2.5A

V_{TH}= 2 (10+I)
V_{TH} = 2 (10-2.5) = 15V





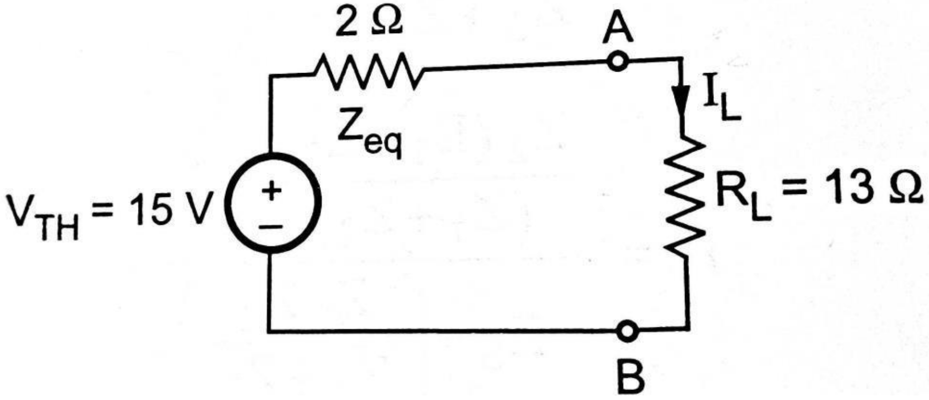
To find equivalent resistance,

R_{eq}= (2||2) +1

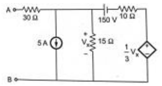
R_{eq}= 1+1 = 2Ω

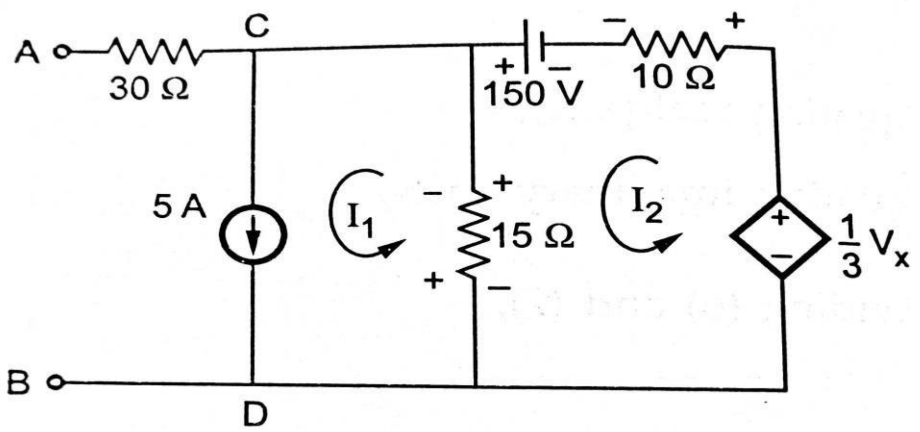
$I_L = \frac{15}{2+13}$

I_L = 1A



Problem5: Calculate Thevenin's equivalent circuit across AB for the network shown in fig.





I_1= 5A ----------------------(1)

Applying KVL to the loop,

$-15I_2$+ 15I_1 +150 - 10I_2 + 1/3 V_x= 0

V_x = 15 (I_2-I_1)

$-15I_2$+ 15I_1 +150 - 10I_2 + 1/3 X 15(I_2-I_1) = 0

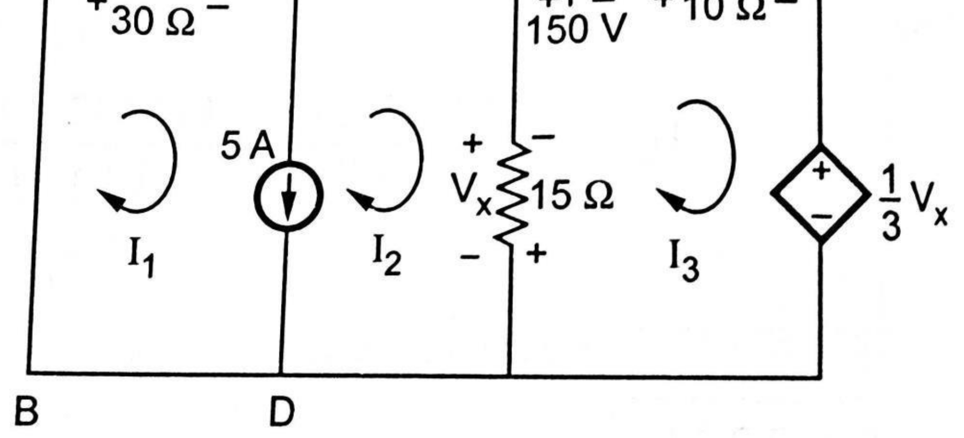
-150 +20I_2 +25-75 = 0

I_2= 10A

V_{TH}= 15 (I_2-I_1)

V_x = V_{TH} = 15 (10-5) = 75V

Due to dependent source, $R_{eq} = \frac{V_{TH}}{I_{SC}}$



I_1-I_2= 5 ----------------------(1)

Applying KVL to super mesh

-30I_1-15I_2+15I_3=0 -----------(2)

Applying KVL to loop,

-150-10I_3-1/3V_x-15I_3+15I_2=0

-150-10I_3-1/3 X 15(I_2-I_3)-15I_3+15I_2=0

-150-10I_3-5I_2 + 5I_3-15I_3+15I_2=0

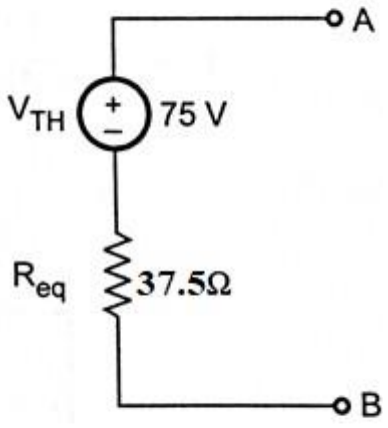
10I_2 - 20I_3 = 150 ----------------------(3)

I_{SC}= -I_1= 2 A

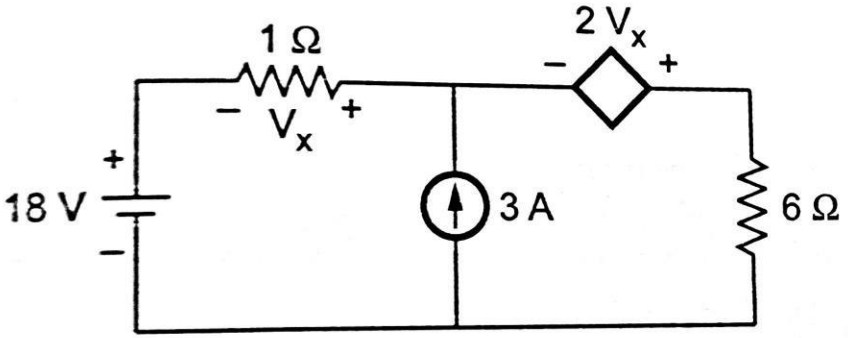
$R_{eq} = \frac{V_{TH}}{I_{SC}}$

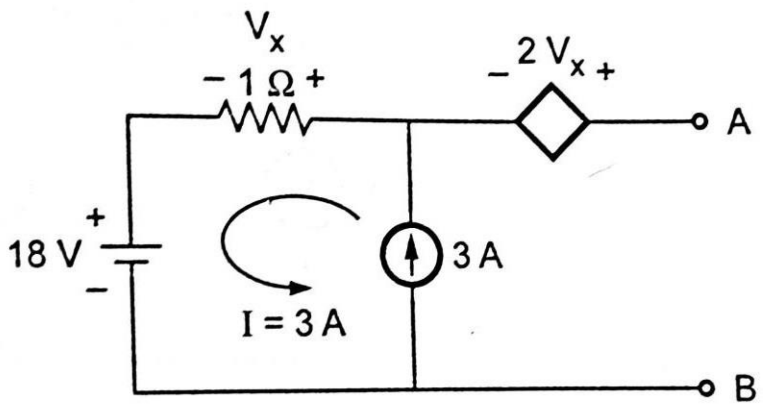
= 75/2

= 37.5Ω



Problem6: **Find Thevenin's voltage, short circuit current and determine the actual current flowing through the 6Ω resistor.**

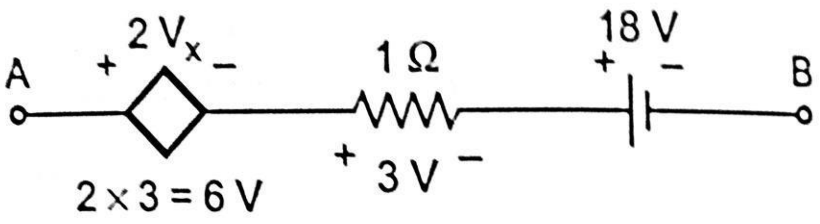




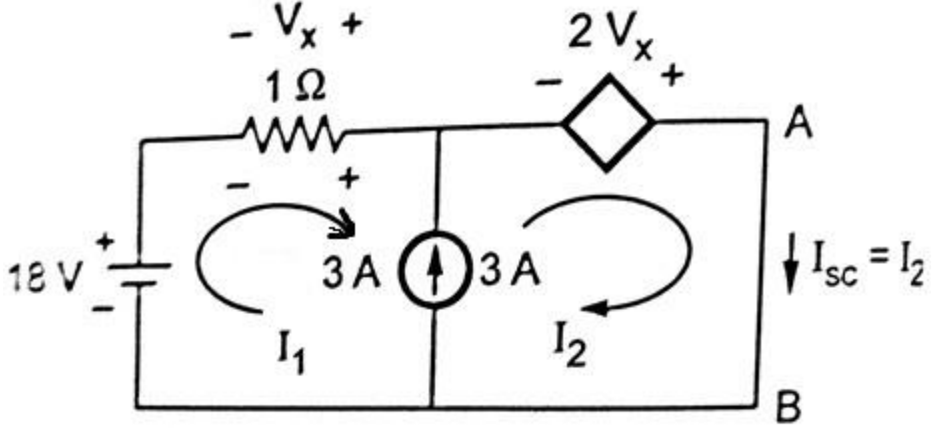
As AB is open, the loop current is 3A.

V_x= 1 X 3 =3V

V_{TH} = 6 + 3 + 18 = 27 V



Due to dependent source, $R_{eq} = \frac{V_{TH}}{I_{SC}}$



I_2-I_1= 3 ----------------------(1)

Applying KVL to super mesh

18 - I_1 + 2V_x= 0

18 - I_1 + 2 X -I_1 = 0

3I_1 = 18

I_1= 6A

I_2= 3 + I_1

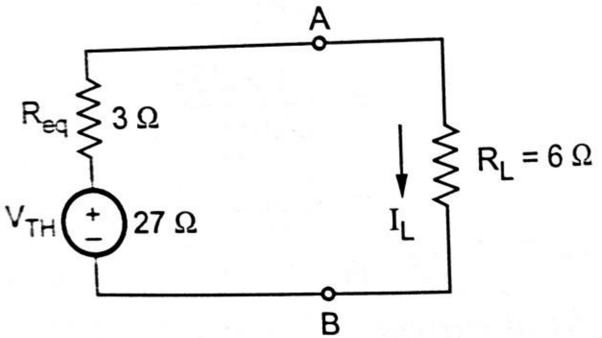
I_2 = 3+6 = 9A

I_{SC} = I_2 = 9A

$R_{eq} = \frac{V_{TH}}{I_{SC}}$

= 75/2

= 37.5Ω

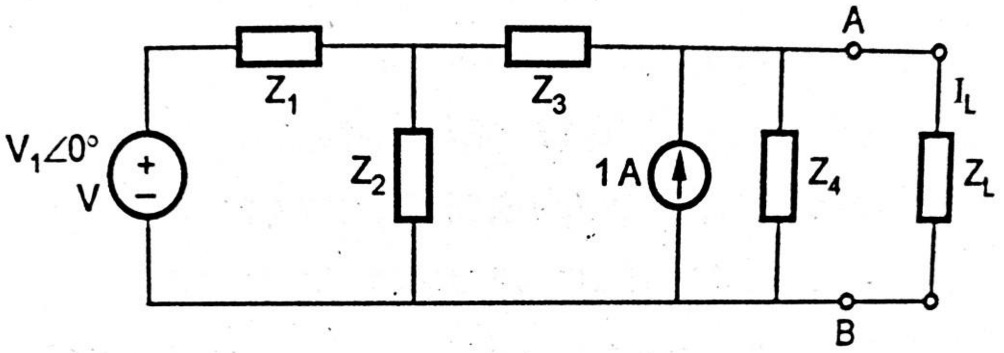


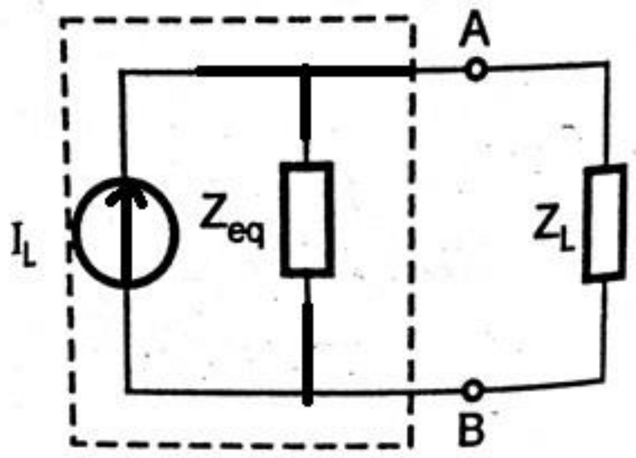
I = $\frac{V_{TH}}{R_L + R_{eq}}$ = $\frac{27}{3+6}$

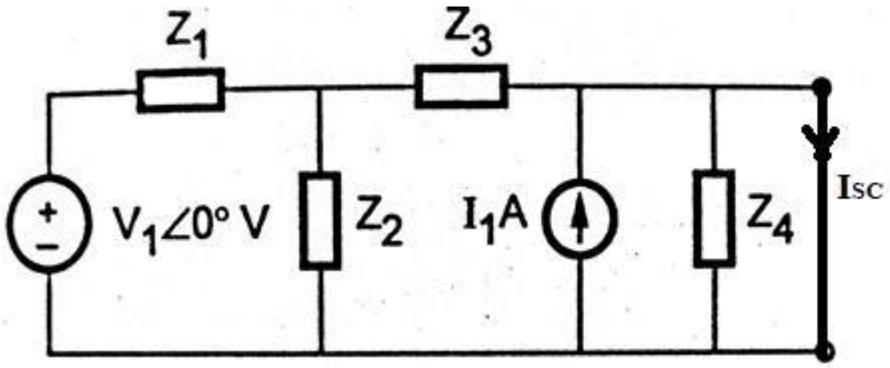
= 3A

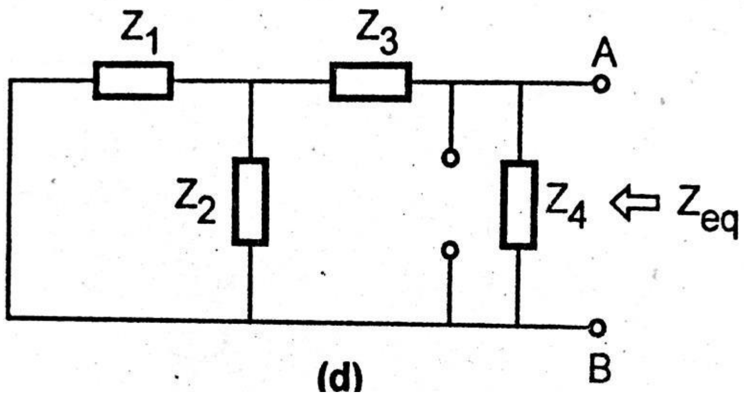
Norton’s Theorem

“Any linear circuit containing several voltages and current sources can be replaced by just one single current source in parallel with a single resistance“.









(d)

Steps to apply Norton’s Theorem

- ✓ **Short the branch, through which current is to be calculated.**

- ✓ **Calculate the current through this short circuited branch, by using any of the network simplification techniques. This current is nothing but Norton’s current I_N.**

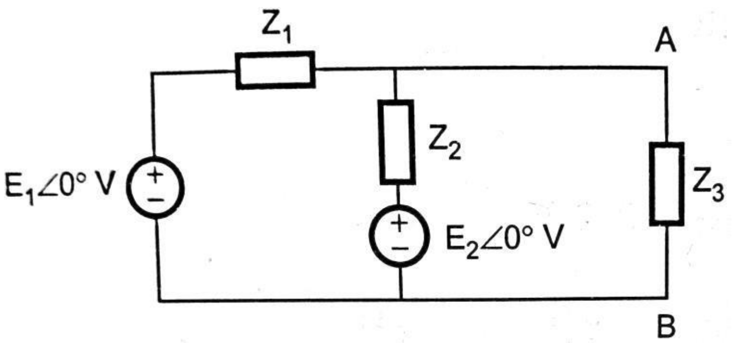
- ✓ **Calculate the equivalent impedance Z_{eq}, as viewed through the two terminals of the branch from which current is to be calculated by removing that branch impedance and replacing all the independent sources by their internal impedances.**

- ✓ **Draw the Norton’s equivalent showing the voltage source I_L, with the impedance Z_{eq} in Parallel with it, across the terminals of interest. Reconnect the branch impedance now. Let it be Z_L. The required current through the branch is given by,**

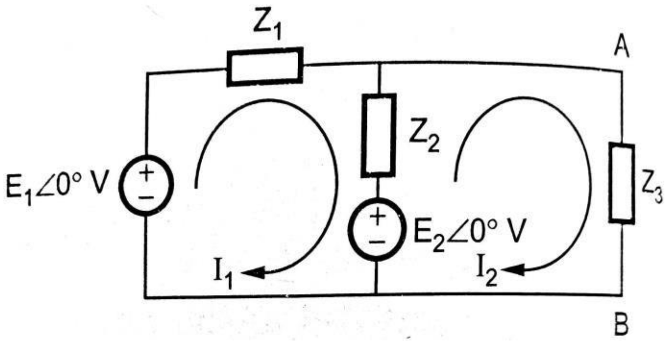
$$\text{I} = I_N \frac{Z_{eq}}{Z_L + Z_{eq}}$$

Proof of Norton's Theorem

Consider the network shown.



Let us find the current through Z_3 by mesh analysis first. Assume the currents as shown.



Applying KVL to the loop,

$-Z_1I_1 - Z_2 I_1 + Z_2 I_2 - E_2 + E_1 = 0$

$(-Z_1 - Z_2)I_1 + Z_2 I_2 = E_2 - E_1$ ------------(1)

$-Z_3I_2 - Z_2I_2 + Z_2I_1 + E_2 = 0$

$Z_2I_1 + (-Z_3 - Z_2)\, I_2 = -E_2$ -----------------(2)

$$D = \begin{vmatrix} -Z_1 - Z_2 & Z_2 \\ Z_2 & -Z_3 - Z_2 \end{vmatrix} = Z_1 Z_2 + Z_2Z_3 + Z_3Z_1$$

Now current through Z_3 is I_2, hence calculating D_2,

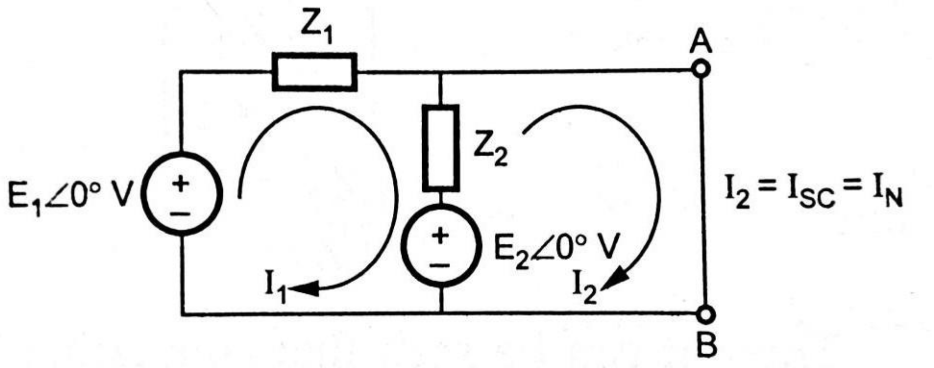
$$D_2 = \begin{vmatrix} -Z_1 - Z_2 & E_2 - E_1 \\ Z_2 & -E_2 \end{vmatrix} = Z_1 E_2 + Z_2 E_1$$

$$I_2 = \frac{D_2}{D}$$

$$= \frac{Z_1E_2 + Z_2E_1}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}$$

Required current

Let us use Norton's Theorem.



Step1: Short the branch A-B.

Step2: Calculate I_N

Applying KVL to two loops

$$-I_1 Z_1 - I_1 Z_2 + I_2 Z_2 - E_2 + E_1 = 0$$

$$I_1(Z_1 + Z_2) - I_2 Z_2 = E_1 - E_1 \text{ ---------------(1)}$$

$$E_2 - I_2 Z_2 - I_1 Z_2 = 0$$

$$I_1 Z_2 - I_2 Z_2 = -E_2 \text{ --------------------------(2)}$$

$$D = \begin{vmatrix} Z_1 + Z_2 & -Z_2 \\ Z_2 & -Z_2 \end{vmatrix} = -Z_1 Z_2$$

with

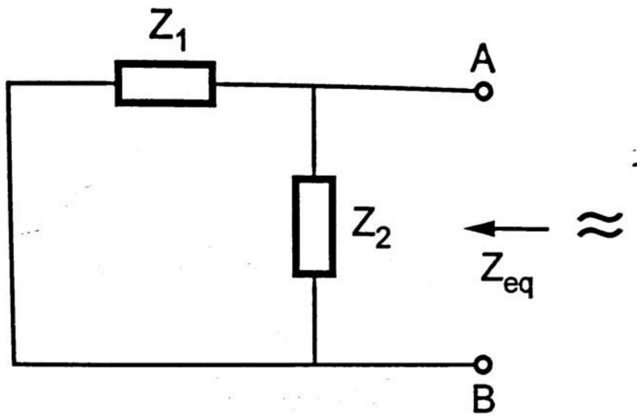
$$D_2 = \begin{vmatrix} Z_1 + Z_2 & E_1 - E_2 \\ Z_2 & -E_2 \end{vmatrix} = -E_2 (Z_1 + Z_2) - Z_2 (E_1 - E_2)$$

$$I_2 = \frac{D_2}{D}$$

$$I_N = \frac{Z_1 E_2 + Z_2 E_1}{Z_1 Z_2}$$

Step3: Obtain Z_{eq} as viewed through terminals A-B, with both the sources replaced by short circuit.

$$Z_{eq} = (Z_1 \,||\, Z_2) = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

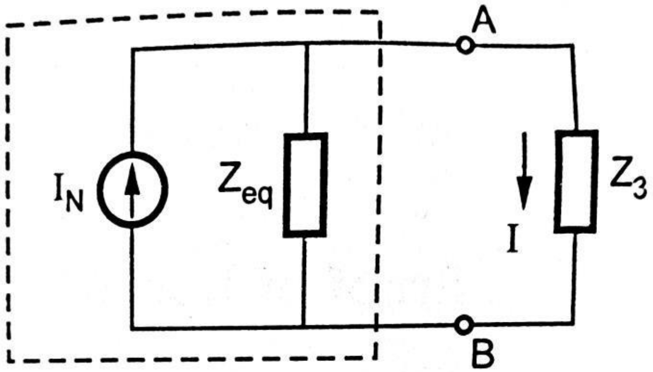


Step4: Norton's equivalent across the terminals AB is as shown. Hence the required current I is,

$$I = I_N \frac{Z_{eq}}{Z_3 + Z_{eq}}$$

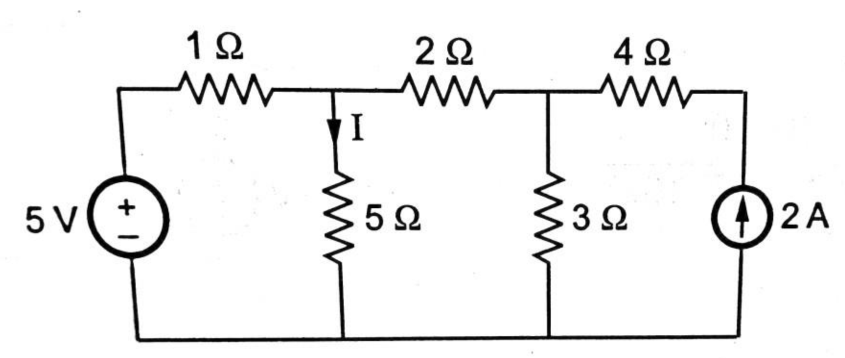
$$= \frac{\frac{Z_1E_2 + Z_2E_1}{Z_1Z_2} X \frac{Z_1Z_2}{Z_1+Z_2}}{Z_3 + \frac{Z_1Z_2}{Z_1+Z_2}}$$

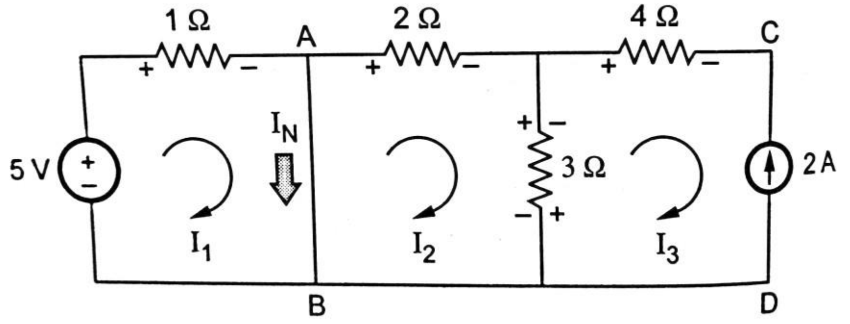
$$I = \frac{Z_1E_2 + Z_2E_1}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}$$



Norton's equivalent

Problem1: Using Norton's Theorem, find the current 'I' of the network shown.





Short the load branch

I_3= -2A ----------------------(1)

Applying KVL to the loop,

-I_1+5 =0

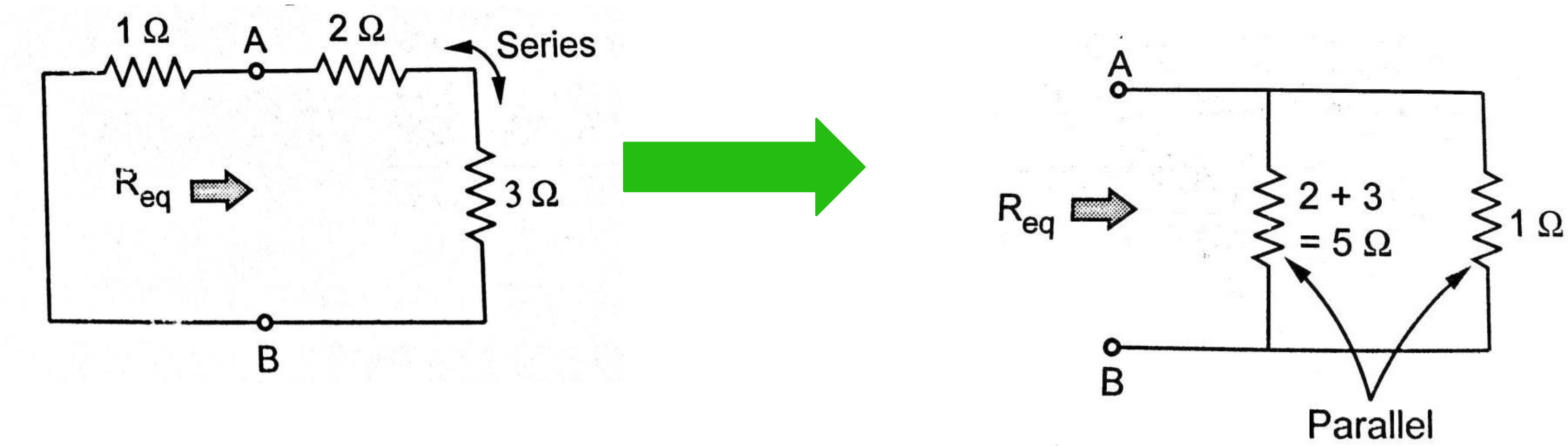
I_1= 5A

$-3I_2$+ $3I_3$ - $2I_2$ = 0

-5I_2+ 3I_3= 0 ➔ 5I_2= 3(-2)

I_2= -6/5 = -1.2A

I_N = I_1- = 5+1.2 = 6.2 A



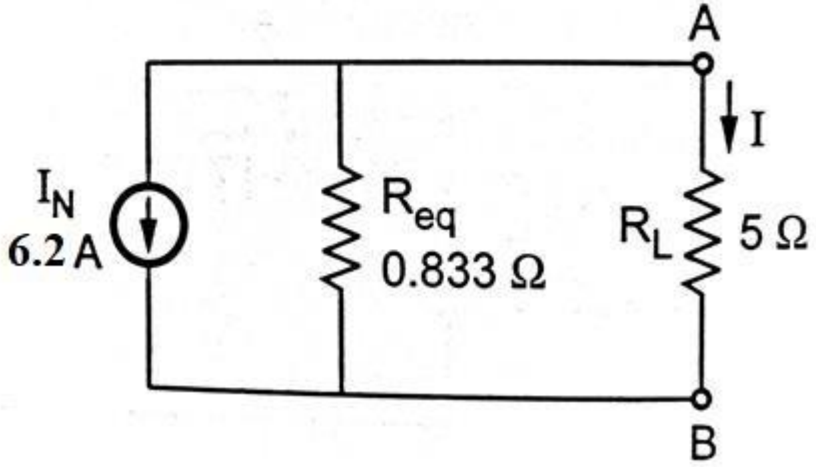
To find equivalent resistance,

R_{eq}= (5||1)

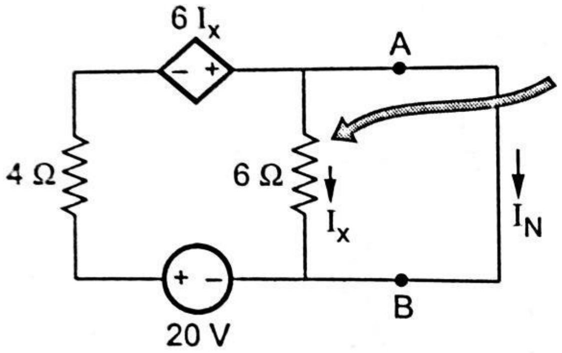
R_{eq}= 0.833Ω

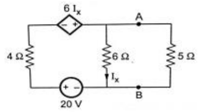
$I_L = I_N \dfrac{R_{eq}}{R_L+R_{eq}}$

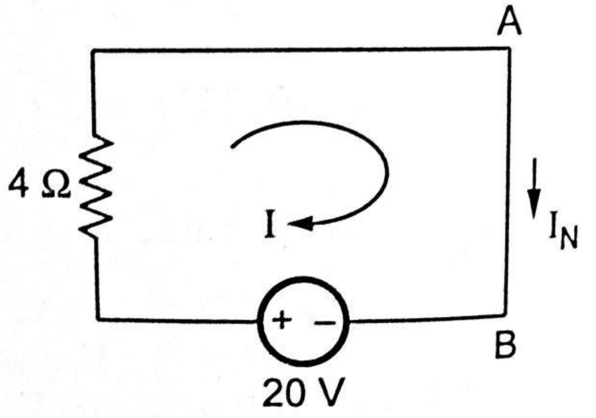
I_L = 0.885A



Problem2: Determine the Norton's equivalent circuit across AB terminals in the the network . Hence determine current in 5Ω resistor. And draw Thevenin's equivalent circuit across AB.





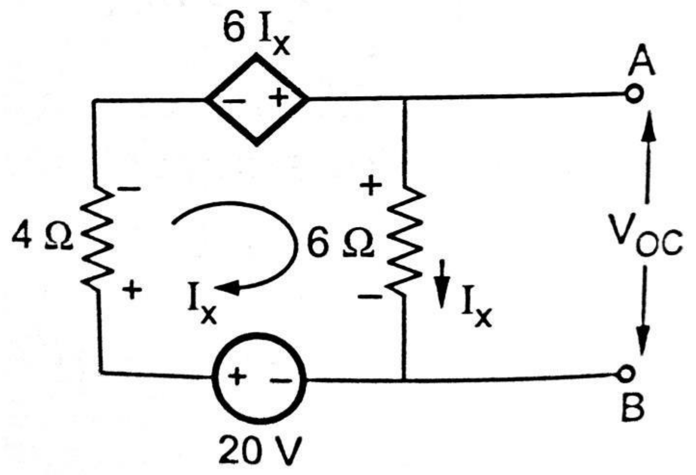


Short the load branch

I_N= 20/4

I_N= 5A

Due to the presence of dependent source



$R_{eq} = \frac{V_{OC}}{I_{SC}}$

To find Voc across AB terminals

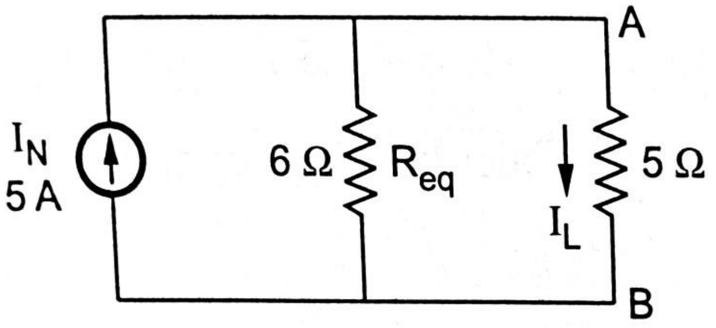
-6 I_x+20 - 4 I_x+6 I_x=0

I_x = 5A

Voc = 6 $I_x = 30V$

$$R_{eq} = \frac{V_{OC}}{I_{SC}}$$

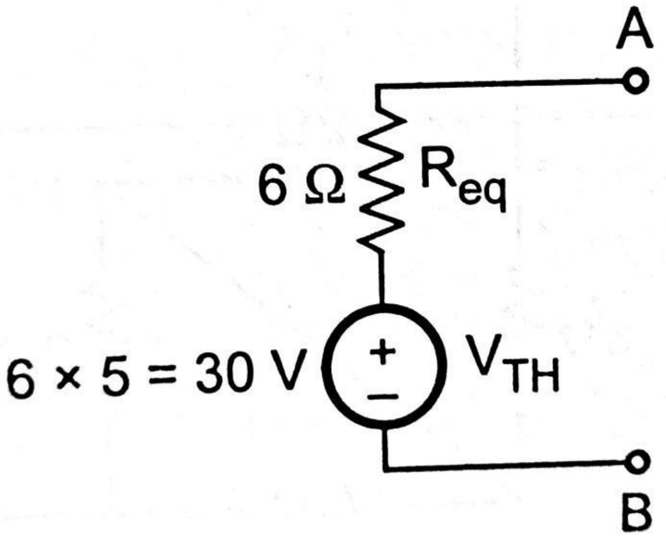
R_{eq} = 30/5 = 6Ω



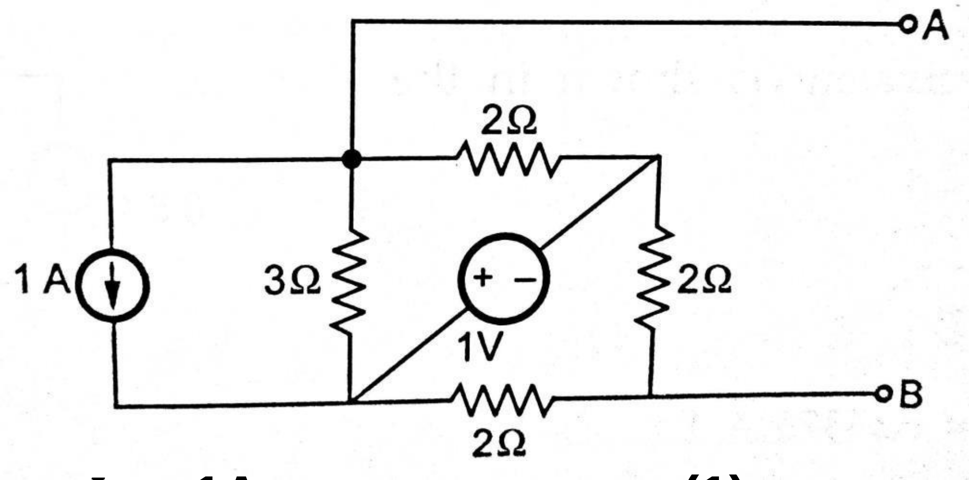
$$I_L = I_N \frac{R_{eq}}{R_L + R_{eq}}$$

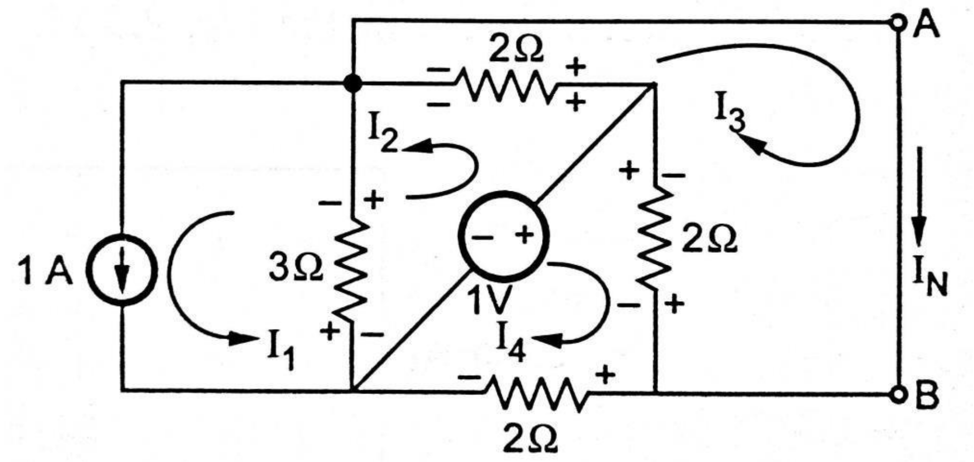
$$I_L = 5 \text{ X } \frac{6}{5+6}$$

$$I_L = 2.7272\text{A}$$



Problem3: Determine the current through 1Ω resistor connected across AB in the network using Norton's Theorem.





I_1 = 1A -------------------------(1)

Applying KVL

-2I_2 - 2I_3 -3I_2 + 3I_1 +1 = 0

-5I_2 - 2I_3 = -4 -------------------------(2)

-2I_4 + 2 I_3 - 2I_4 +1 = 0

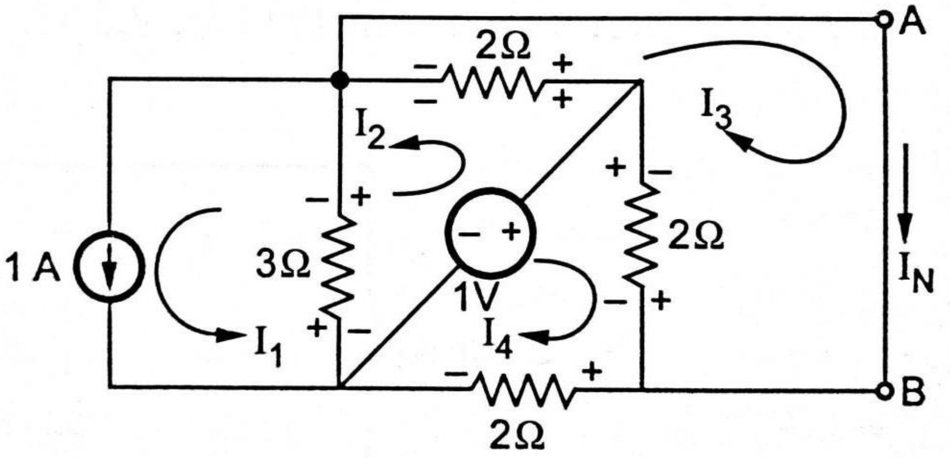
2 I_3 - 4I_4 = -1 -----------------------(3)

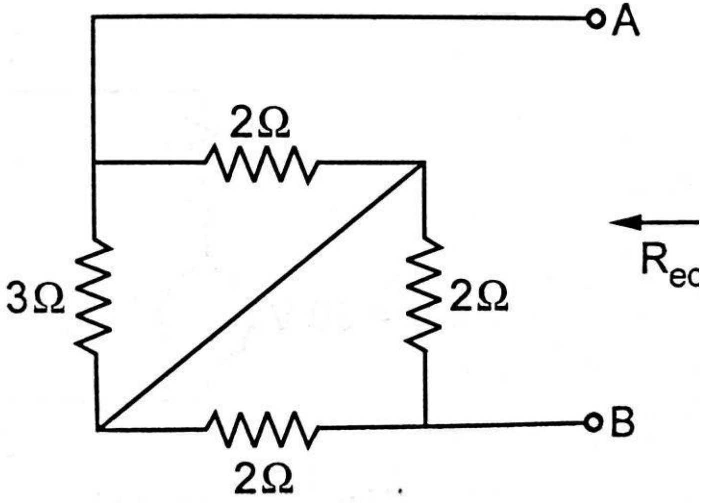
-2I_3+2I_4-2I_3-2I_2=0

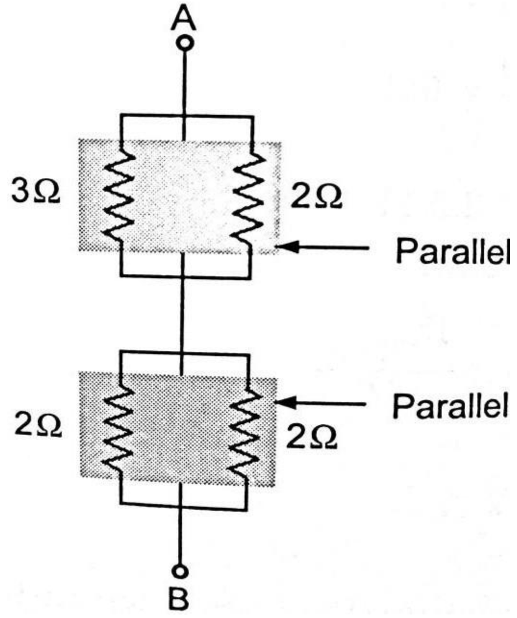
-2I_2-4I_3+2I_4 =0 ----------------------(4)

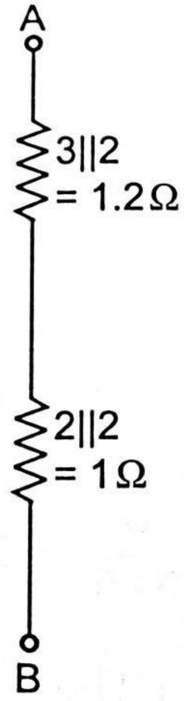
I_3 = 1 A, I_3 = -0.5 A, I_3 = 0 A

I_N = -0.5 A

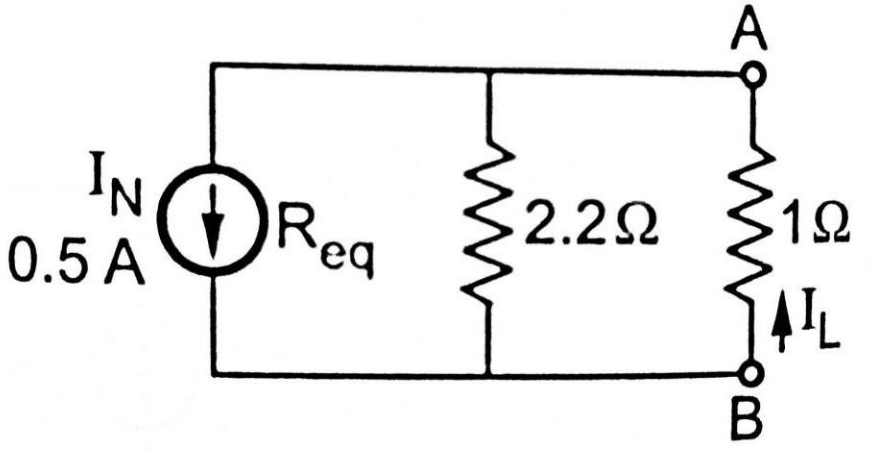

A
2Ω
I_3
I_2
1 A
3Ω
1V
2Ω
I_N
I_1
I_4
B
2Ω


A
2Ω
3Ω
2Ω
R_{eq}
B
2Ω


A
3Ω
2Ω
Parallel
2Ω
2Ω
Parallel
B


A
3||2
= 1.2Ω
2||2
= 1Ω
B

$R_{eq} = 2.2\ \Omega$

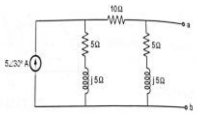


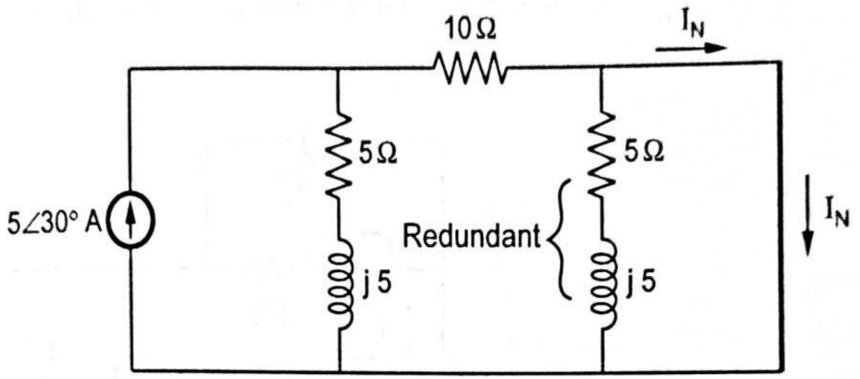
$$I_L = I_N \frac{R_{eq}}{R_L+R_{eq}}$$

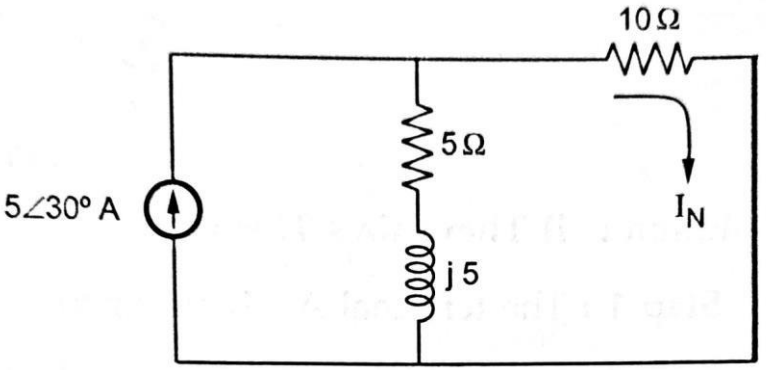
$$I_L = 0.5 \text{ X } \frac{2.2}{1+2.2}$$

$$I_L = 0.34375\text{A}$$

Problem4: Obtain Thevenin's and Norton's equivalent circuits across the terminals a-b of the circuit shown.

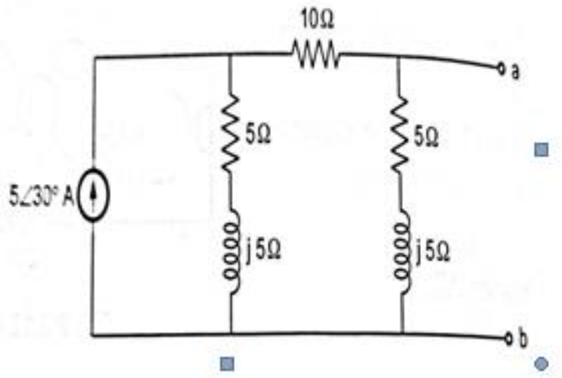


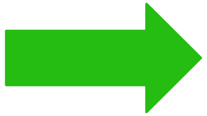


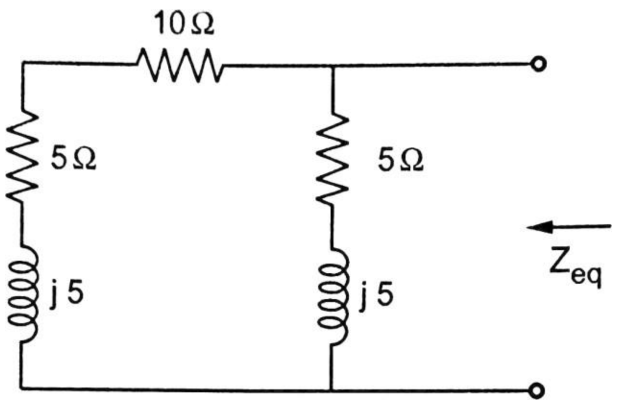


$$I_N = 5\angle 30 \, \frac{(5+5j)}{10+(5+5j)}$$

$$I_N = 2.2361\angle\, 56.57 \text{ A}$$



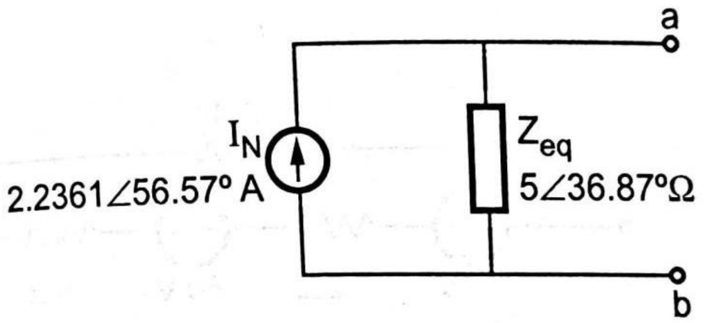




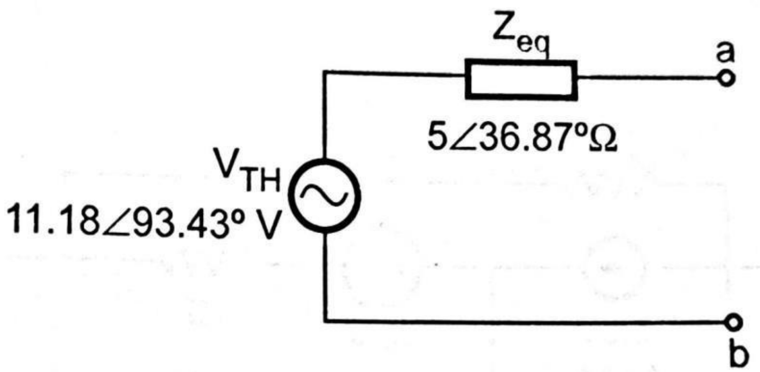
R_{eq} = (5+5j) || (15+5j)

$$= \frac{(5+5j)(15+5j)}{(5+5j)+(15+5j)}$$

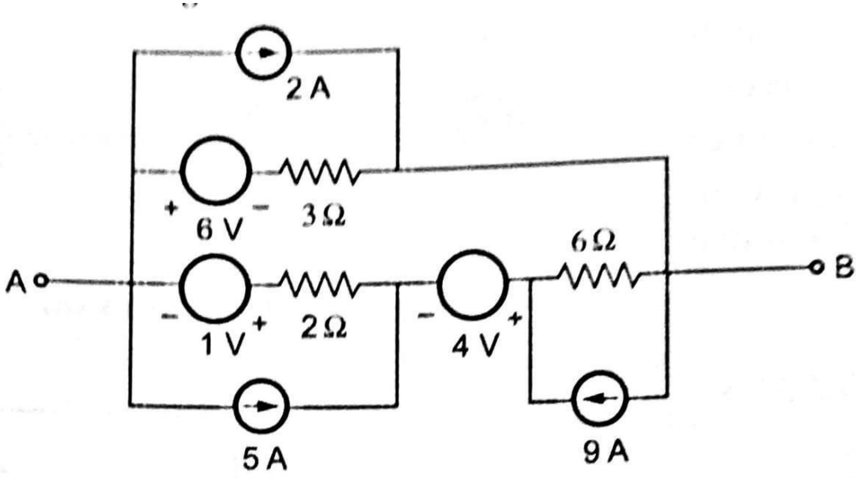
R_{eq} = 5 ∠ 36.86 Ω

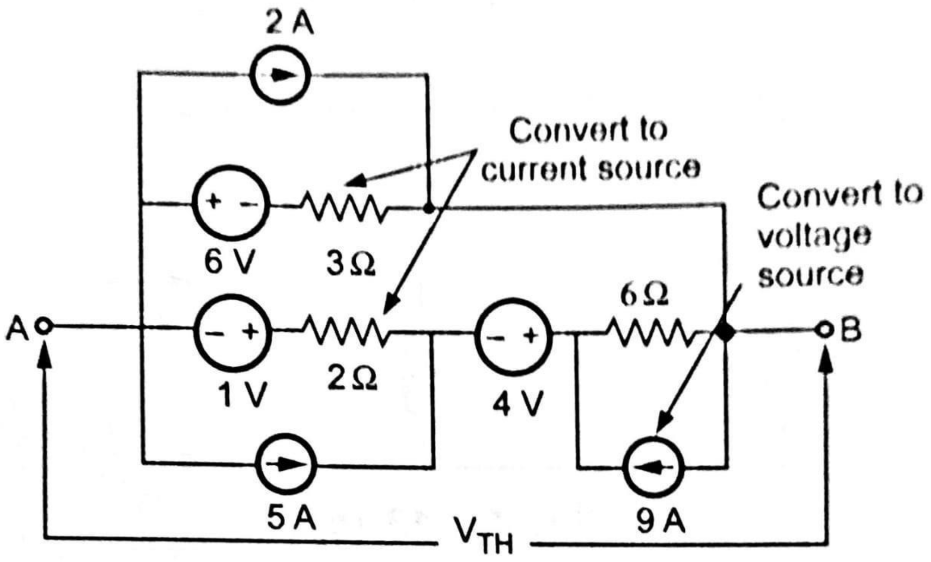


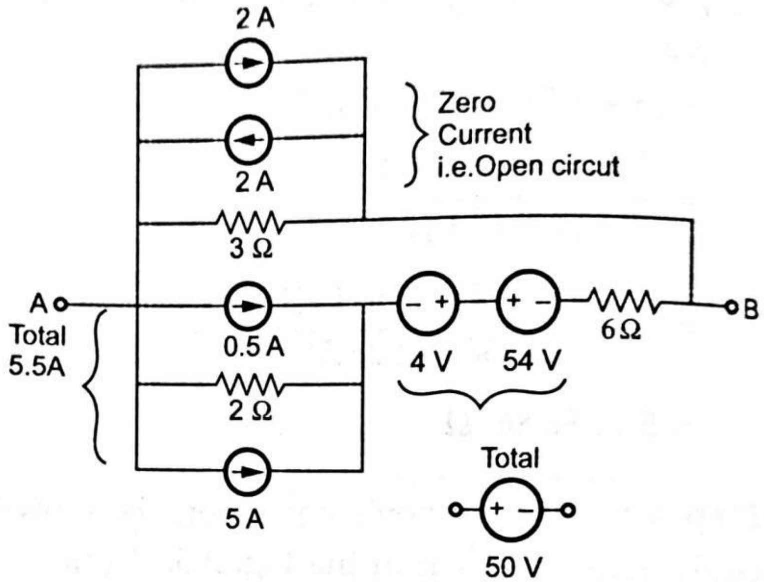
$V_{TH} = I_N Z_{eq}$ = 11.1805∠93.43 V

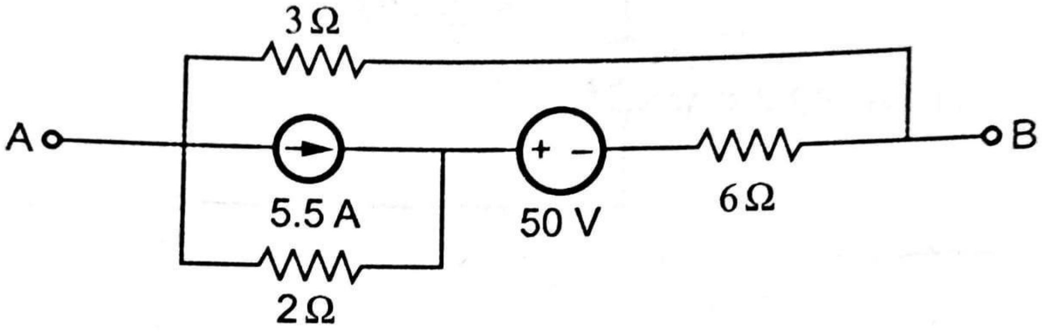


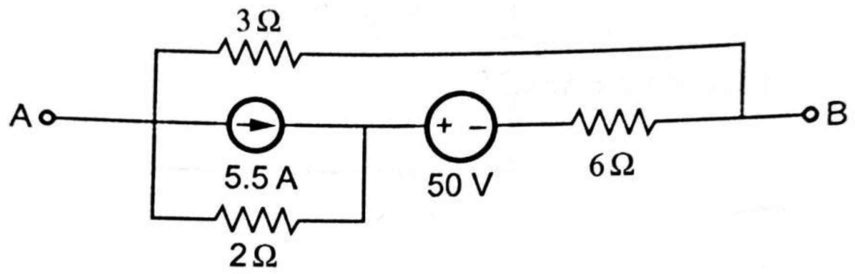
Problem5: Obtain Thevenin's and Norton's equivalent circuits across the terminals A and B for the circuit shown.

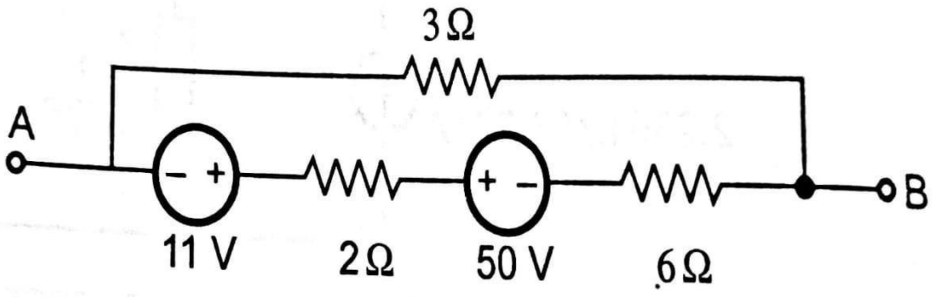












Applying KVL

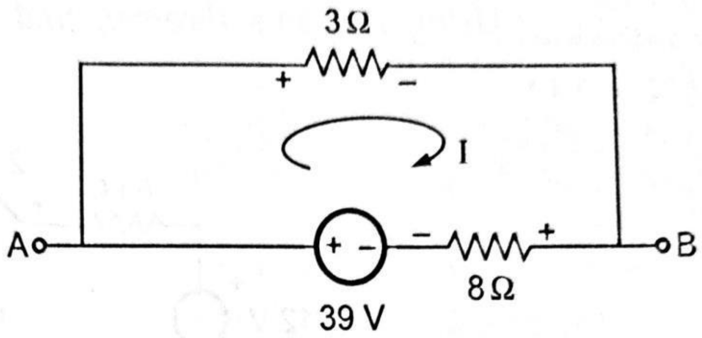
-3I - 8I + 39 = 0

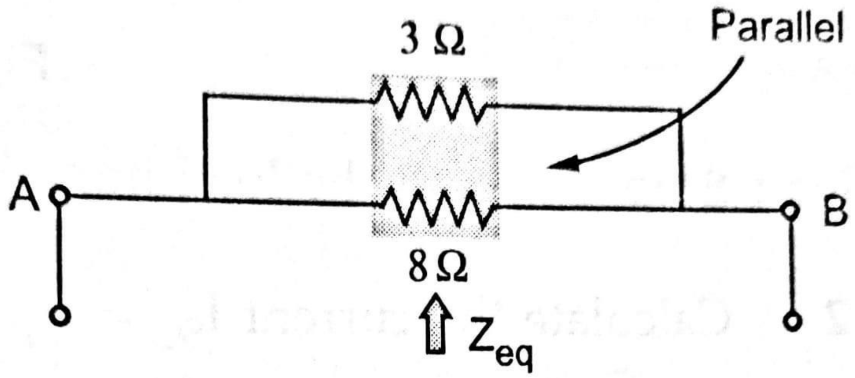
-11 I +39 = 0

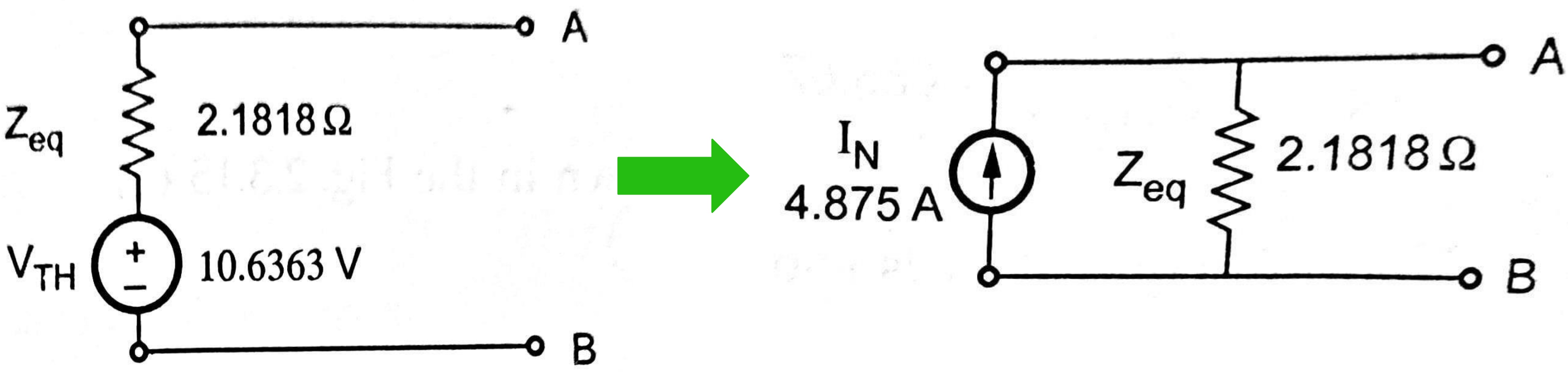
I = 39/11 = 3.5454A

V_{TH} = 3 I = 10.6363V

R_{eq} = 3||8 = 2.1818 Ω



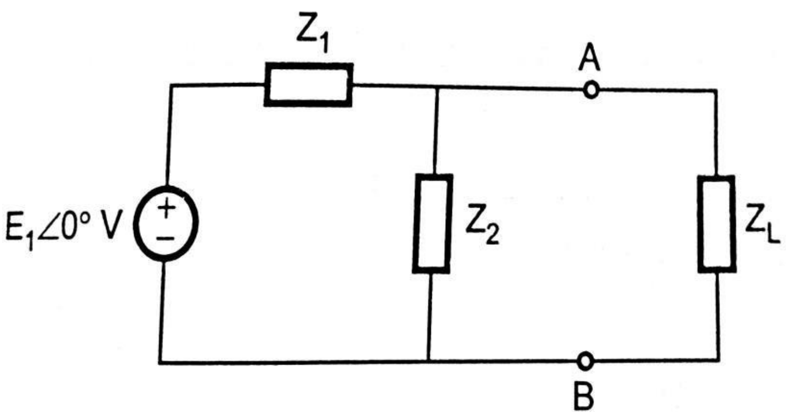


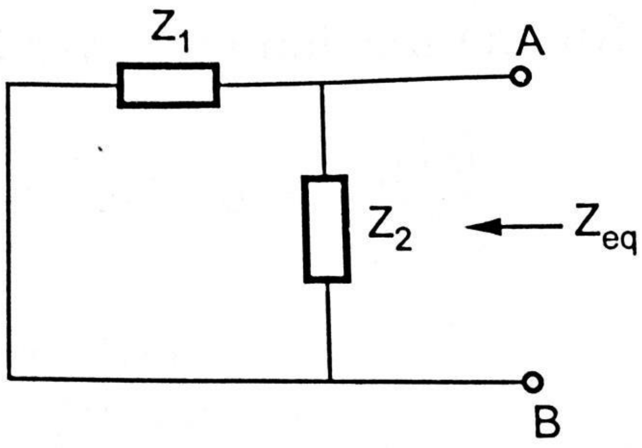

A
Z_{eq}
2.1818 Ω
V_{TH}
10.6363 V
B
I_N
4.875 A
Z_{eq}
2.1818 Ω
A
B

Maximum Power Transfer Theorem

"Maximum power is transferred from the source to the load when the load resistance is equal to the thevenin's equivalent resistance."

Let Z_{eq} be the equivalent impedance of the network as viewed from the terminals A-B and replacing all the independent sources by their internal impedances, as shown in the figure.





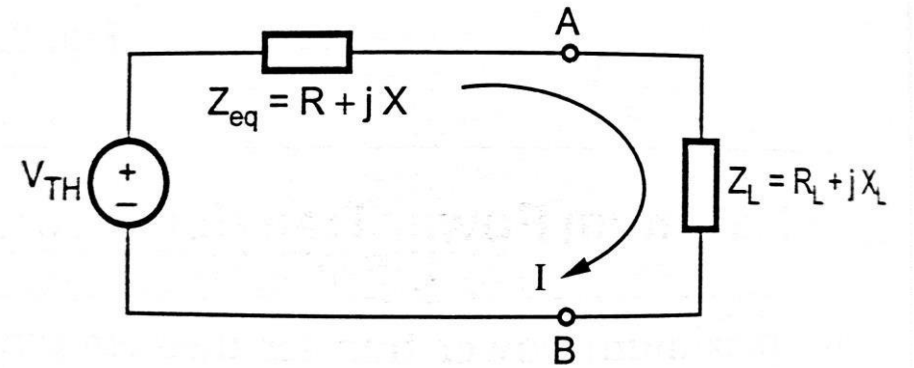
Let Z_{eq}= R + j X

Then the maximum power will be transferred to the load, if Z_L is complex conjugate of Z_{eq}.

$Z_L = Z_{eq}^* =$ R - jX

Proof of Maximum Power Transfer Theorem

Let the given network is replaced by its Thevenin's equivalent across the load terminals as shown.



Let Z_{eq}= R + jX Ω

And Z_L= R_L + jX_L Ω

I = $\frac{V_{TH}}{Z_{eq}+Z_L}$ = $\frac{V_{TH}}{R + jX + R_L + jX_L}$

I = $\frac{V_{TH}}{(R + R_L) + j\,(X_L + X)}$

The power delivered to the load is $P_L = I^2 R_L$

Now the magnitude of the current is, I = $\frac{VTH}{\sqrt{(R + R_L)^2 + (X_L + X)^2}}$

$$P_L = \frac{{V_{TH}}^2}{(R + R_L) + (X_L + X)^2} \cdot R_L$$

Now for the load impedance Z_L both R_L and X_L are variable and are to be decided such that power will be maximum. Hence according to maxima theorem, we can write that for maximum power transfer, with respect to variable X_L and fixed R_L.

$$\frac{\partial P_L}{\partial X_L} = 0$$

$$P_L = \frac{V_{TH}^2}{(R + R_L) + (X_L + X)^2} \cdot R_L$$

$$\frac{\partial}{\partial}\left[\frac{V_{TH}^2 R_L}{(R + R_L)^2 + (X_L + X)^2}\right] = 0$$

$$\frac{-2V_{TH}^2 R_L (X_L + X)}{[(R + R_L) + (X_L + X)^2]^2} = 0$$

$X + X_L = 0 \quad \Rightarrow \quad X_L = -X$

Thus load reactance must be same in magnitude of the reactance of Z_{eq} but opposite in sign.

Similarly power transfer will be maximum with respect to variable R_L and fixed X_L

$\frac{\partial P_L}{\partial X_L} = 0$ i.e $\frac{\partial}{\partial}\left[\frac{V_{TH}^2 R_L}{(R + R_L)^2 + (X_L + X)^2}\right] = 0$

Substituting X_L= -X, as already derived for maximum power,

$$\frac{\partial}{\partial}\left[\frac{V_{TH}^2 R_L}{(R + R_L)^2}\right] = 0$$

$$\frac{(R + R_L)^2 V_{TH}^2 - V_{TH}^2 R_L\, 2(R + R_L)}{(R + R_L)^4} = 0$$

$$(R + R_L)^2 V_{TH}^2 - V_{TH}^2 R_L\, 2(R + R_L) = 0$$

$$(R + R_L)^2 - R_L\, 2(R + R_L) = 0$$

$$R^2 + 2RR_L + R_L^2 - 2RR_L - 2R_L^2 = 0$$

$$R^2 = R_L^2$$

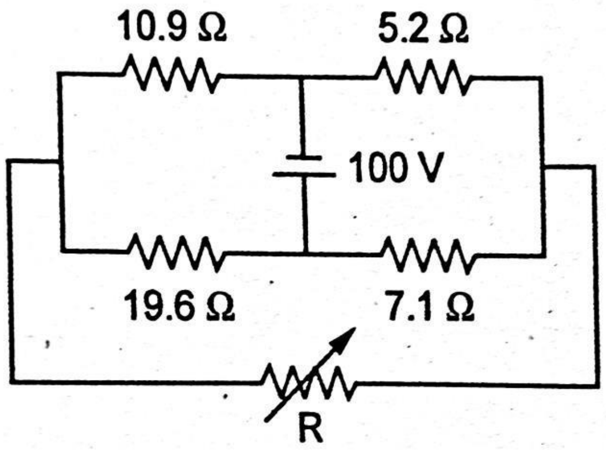
$$R = R_L$$

Thus the resistance of load must be same as that of equivalent impedance of the network. Thus when Z_L is complex conjugate of Z_{eq}, The power transfer to the load is maximum and is given by,

$$P_{max} = I^2 \mathrm{R} = \frac{{V_{TH}}^2 R_L}{(2R_L)^2}$$

$$P_{max} = \frac{{V_{TH}}^2}{4R_L}$$

Problem1: For the circuit shown, find the value of R that will receive maximum power. Determine this power.



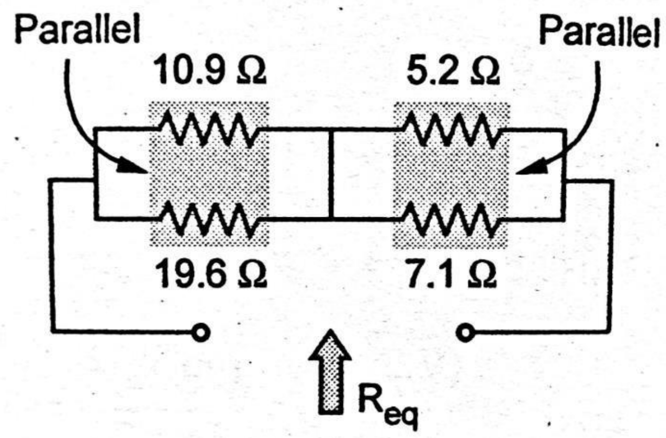
To find P_{max}, R = R_{eq} obtained by opening the load branch and shorting voltage source as

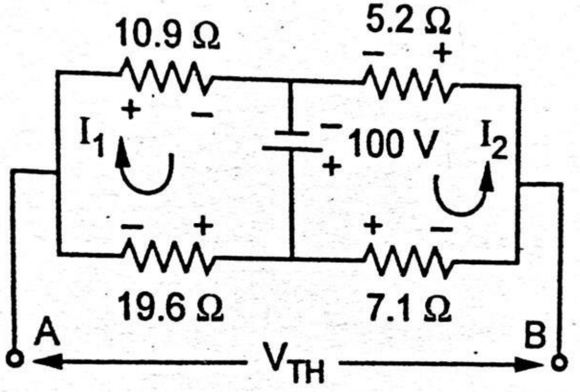
R_{eq}= (10.9||19.6) + (5.2||7.1)

= 7.0046 + 3.0016

R_{eq}= 10.0062 Ω

R = R_{eq} for P_{max}





$-10.9\, I_1 + 100 - 19.6\, I_1 = 0$

I_1 = 3.2786 A

$-5.2\, I_2 + 100 - 7.1\, I_2 = 0$

I_2 = 8.13 A

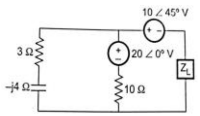
V_{TH} = (19.6X3.2786) – (7.1X8.13)

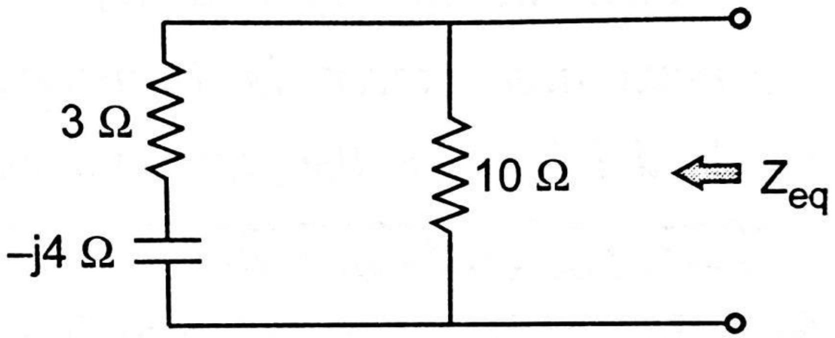
V_{TH} = 6.5375 V

$$P_{max} = \frac{V_{TH}{}^2}{4R}$$

$$P_{max} = \frac{(6.5393)^2}{4X10.0062} = 1.0684 \text{ W}$$

Problem2: Find the value of Z_L for which maximum power transfer occurs in the circuit given.





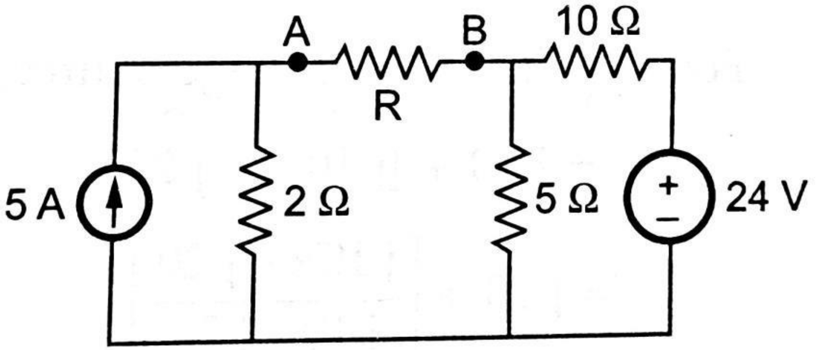
Z_{eq}= 10 || (3-4j)

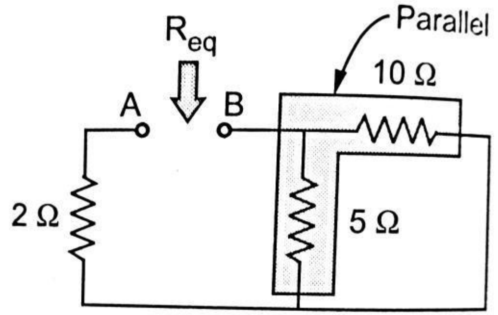
$$= \frac{10\ X\ (3-4j)}{10+3-4j}$$

Z_{eq}= 2.973-2.1622j Ω

R_L = $Z_{eq}{}^{*}$ = 2.973 + 2.1622j Ω for Pmax

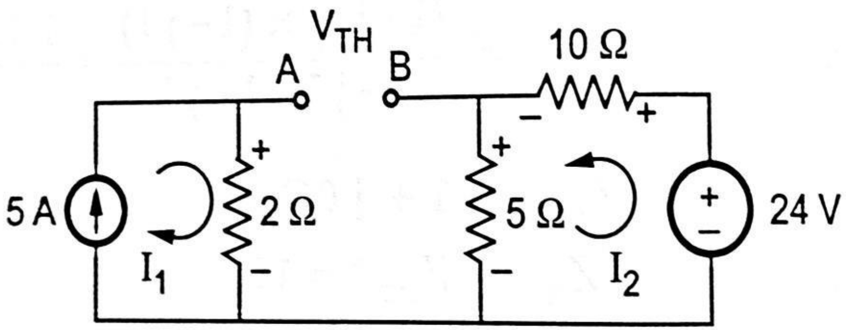
Problem3: For the circuit shown, what would be the value of R such that maximum power transfer can takes place from the rest of the network to ‘R’. Obtain the amount of power.

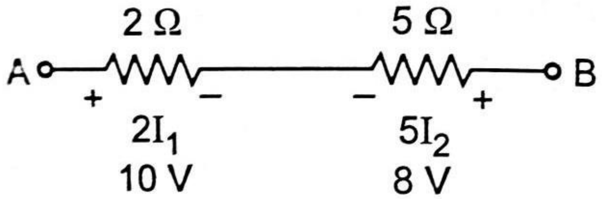




R_{eq}= 2 + (10 || 5)

= 5.33 Ω





I_1 = 5 A

$I_2 = \frac{24}{10+5}$

I_2 = 1.6 A

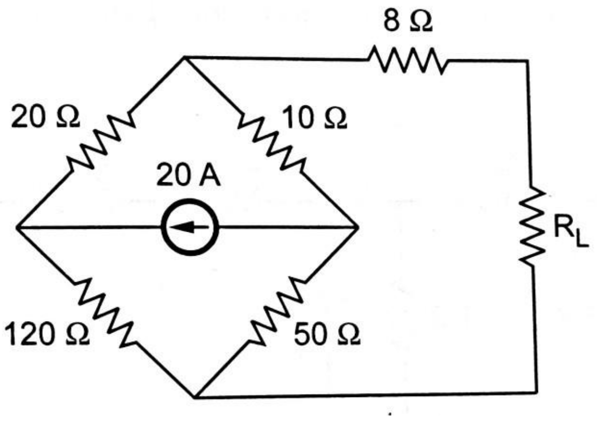
V_{TH} = (2X5) – (5X1.6)

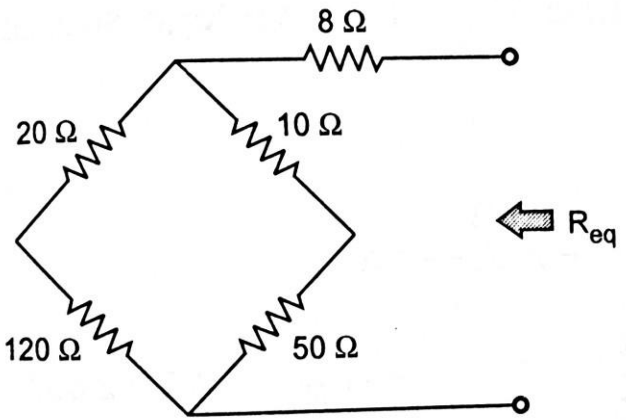
V_{TH} = 10-8 = 2 V

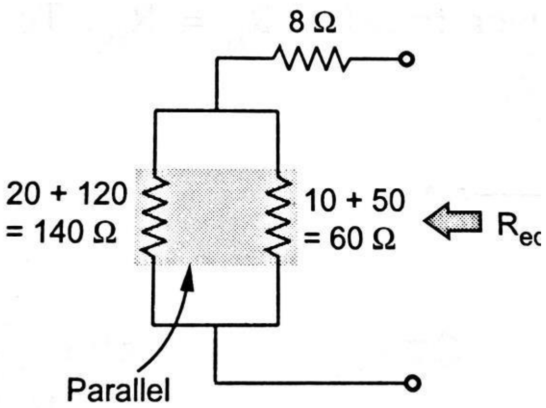
$P_{max} = \frac{{V_{TH}}^2}{4R}$

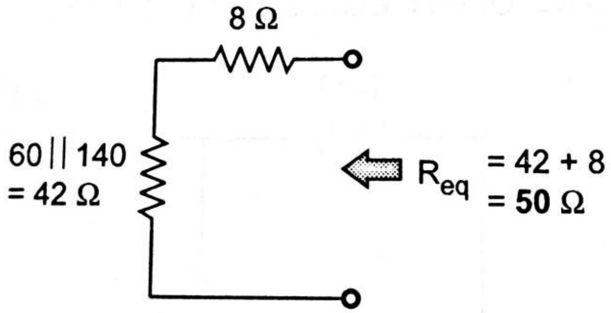
$P_{max} = \frac{(2)^2}{4X5.33}$ = 0.1875 W

Problem4: In the circuit shown, Find the value of R_L for which maximum power is delivered. Also find the maximum power that is delivered to the load.





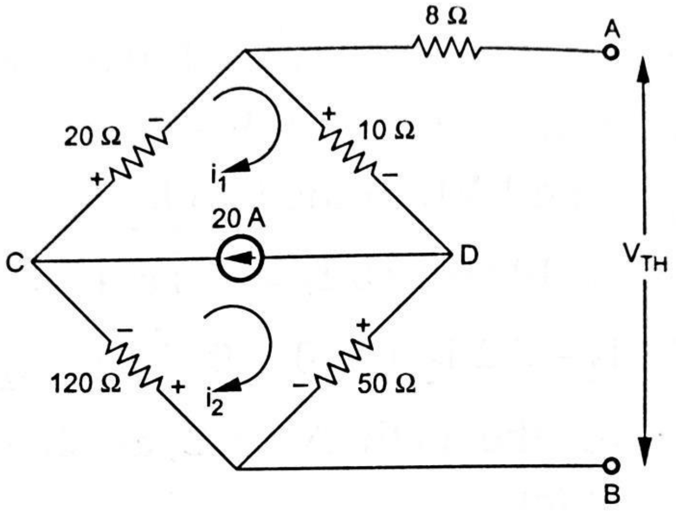




R_{eq}= 42 + 8

R_{eq}= 50 Ω

R_L= R_{eq} for P_{max}



I_1 - I_2 = 20 ----------------------(1)

-20I_1 - 10I_1 - 50I_2 – 120 I_2 = 0
-30I_1 - 170I_2 = 0 ---------------(2)

I_1 = 17 A

I_2 = -3 A

V_{TH} = (10X17) + (50X-3)
V_{TH} = 170 - 150
V_{TH} = 20 V

$$P_{max} = \frac{V_{TH}^{2}}{4R}$$

$$P_{max} = \frac{(20)^2}{4X50} = 2\ \text{W}$$

MODULE-5
TWO PORT NETWORK PARAMETERS

By
K PRABHAVATHI
Assistant Professor
Department of ECE
BGSIT, BG Nagara

Contents

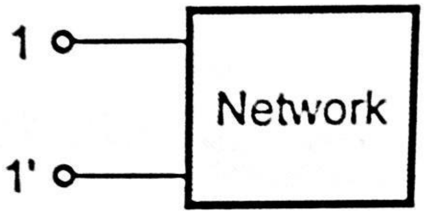
- Definition of z, y, h and Transmission parameters
- Modelling with these parameters
- Relationship between parameters sets

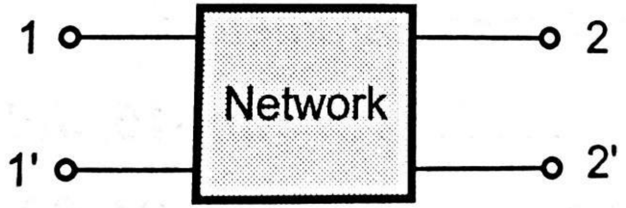
Introduction

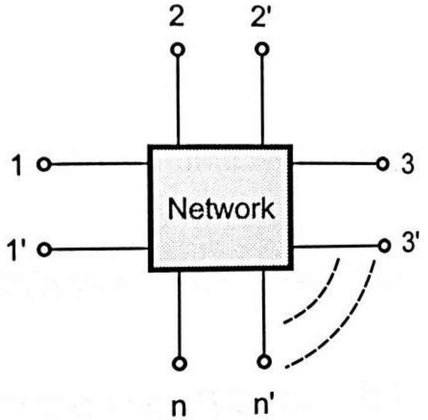
- **Port:** A pair of terminals at which an electrical signal may enter or leave a network.

- **Why ports are required?**
  - To connect input excitation to the network.
  - To connect load
  - To make measurements.

- **Types of network**
  - One port network
  - Two port network
  - Multiport network

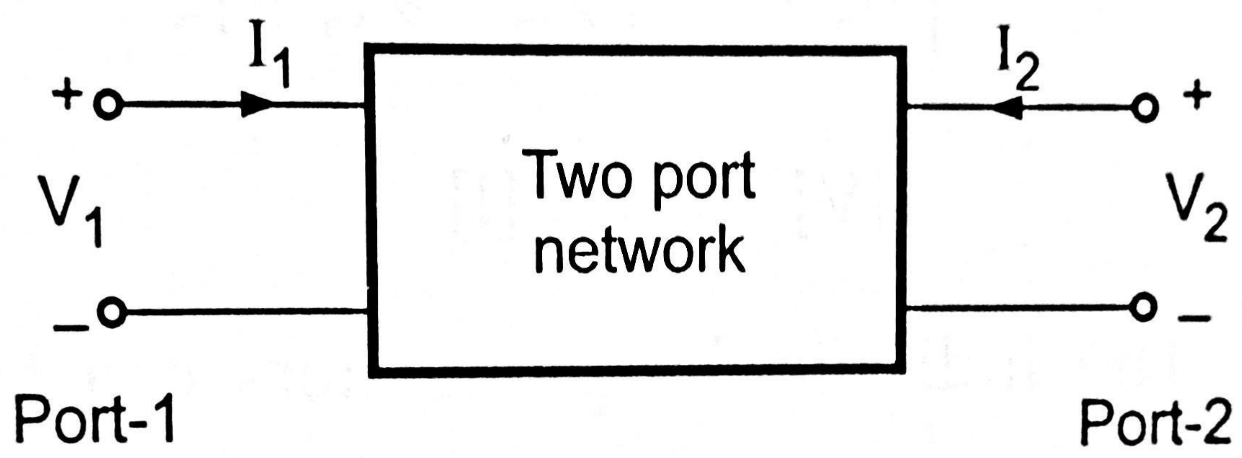






Two Port Network

- **Two Port Network**
  - **Driving port/input port:** at which energy source is connected.
  - **Output port:** at which load is connected.



- **Four variables:** V_1, V_2, I_1 *and* I_2
  - Any two port network can be described with these four variables
  - Network can be considered as black box for analysis without knowing network details.

Two Port Network

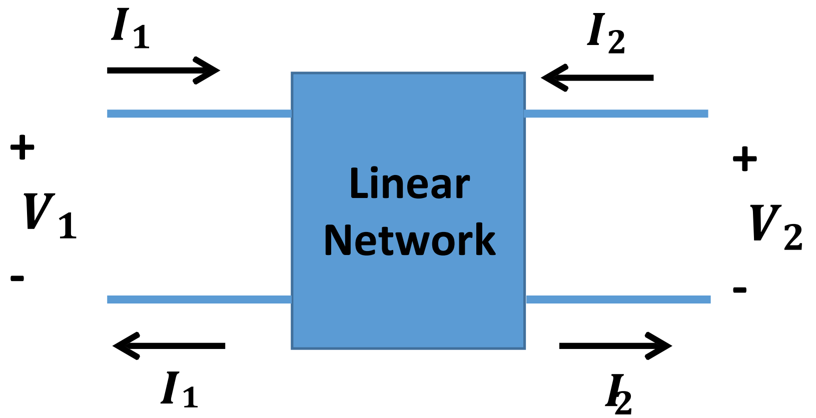
- **Assumptions**
  - Measurements can be made on the box consisting network using only port variables.
  - The network inside the box is assumed to consist only the linear elements.
  - The network may consist of dependent sources but independent sources are not allowed.
  - If the network consists of energy storing elements such as inductor and capacitor then the initial condition on them is assumed to be zero.

- **For Network analysis:**
  - Four variables are represented in terms of two linear equations.
  - Two variables are considered as dependent while other two are considered as independent variables.

- **Six possible ways of selecting two independent variables:**
  - z parameters (Open Circuit Impedance Parameters)
  - y parameters (Short Circuit Admittance Parameters)
  - h parameters (Hybrid Parameters)
  - g parameters (Inverse Hybrid Parameters)
  - ABCD parameters (Transmission Parameters)
  - A’B’C’D’ (Inverse Transmission Parameters)

z Parameters or Open Circuit Impedance Parameters

- They are obtained by expressing voltages at two ports in terms of currents at two ports.
- $\boldsymbol{I_1} \, and \, \boldsymbol{I_2}$ are independent variables
- $\boldsymbol{V_1} \, and \, \boldsymbol{V_2}$ are dependent variables



$$\boldsymbol{V_1 = f_1\,(I_1\,,I_2\,)}$$
$$\boldsymbol{V_2 = f_2\,(I_1\,,I_2\,)}$$

Above equations and be written as

$$\boldsymbol{V_1 = z_{11}\,I_1 + z_{12}I_2 \quad \ldots..eq\;(1)}$$
$$\boldsymbol{V_2 = z_{21}\,I_1 + z_{22}I_2 \quad \ldots..eq\;(2)}$$

In matrix form, equations can be written as

$$\begin{matrix} \boldsymbol{V_1} \\ \boldsymbol{V_2} \end{matrix} = \begin{matrix} \boldsymbol{z_{11}} & \boldsymbol{z_{12}} \\ \boldsymbol{z_{21}} & \boldsymbol{z_{22}} \end{matrix} \quad \begin{matrix} \boldsymbol{I_1} \\ \boldsymbol{I_2} \end{matrix}$$

$$\boldsymbol{V = z \quad I}$$

z Parameters or Open Circuit Impedance Parameters

- The individual z Parameters can be obtained as follows

$V_1 = z_{11} I_1 + z_{12} I_2$

$V_2 = z_{21} I_1 + z_{22} I_2$

1. To obtain z_{11}

$Let\ I_2 = 0 \rightarrow Port\ 2\ is\ open\ circuited$

From eq (1) $V_1 = Z_{11} I_1$

$$\boxed{z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \Omega}$$

The parameter Z_{11} is called open circuit driving point input impedance.

2. To obtain z_{21}

$Let\ I_2 = 0 \rightarrow Port\ 2\ is\ open\ circuited$

From eq (2) $V_2 = Z_{21} I_1$

$$\boxed{z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} \Omega}$$

The parameter Z_{21} is called open circuit forward transfer impedance.

z Parameters or Open Circuit Impedance Parameters

$$V_1 = z_{11} I_1 + z_{12} I_2$$

$$V_2 = z_{21} I_1 + z_{22} I_2$$

3. To obtain z_{12}

$$Let\ I_1 = 0 \rightarrow Port\ 1\ is\ open\ circuited$$

From eq (1) $\quad V_1 = Z_{12}\, I_2$

$$\boxed{z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \Omega}$$

The parameter Z_{21} is called open circuit reverse transfer impedance.

4. To obtain z_{22}

$$Let\ I_1 = 0 \rightarrow Port\ 1\ is\ open\ circuited$$

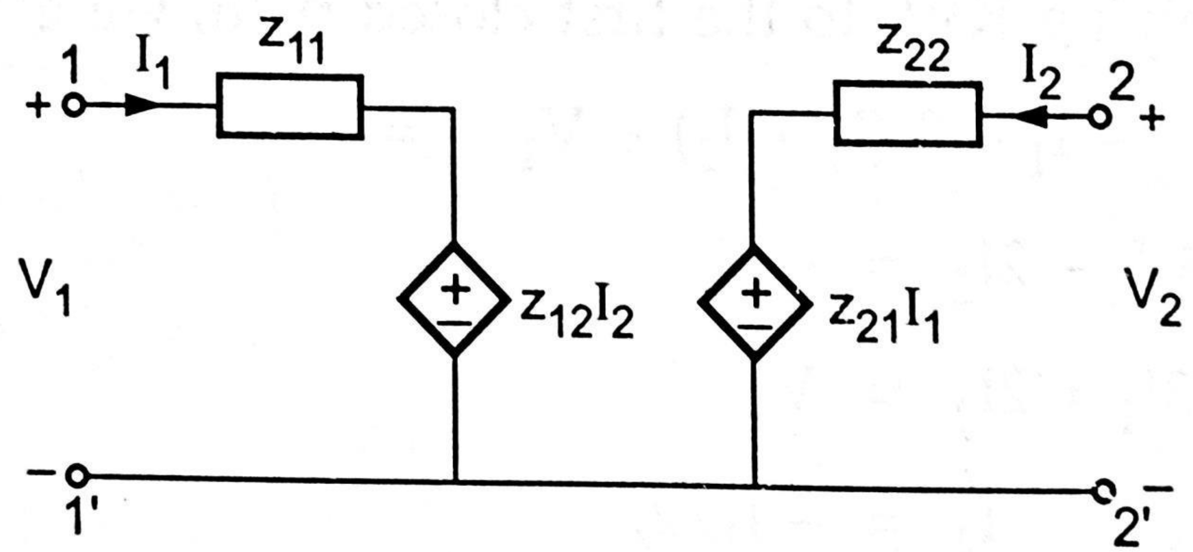
From eq (2) $\quad V_2 = Z_{22}\, I_2$

$$\boxed{z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} \Omega}$$

The parameter Z_{22} is called open circuit driving point output impedance.

z Parameters or Open Circuit Impedance Parameters

- **Equivalent Network in terms of z parameters**



- **Conditions for Symmetry**

$$z_{11} = z_{22}$$

- **Conditions for Reciprocity**

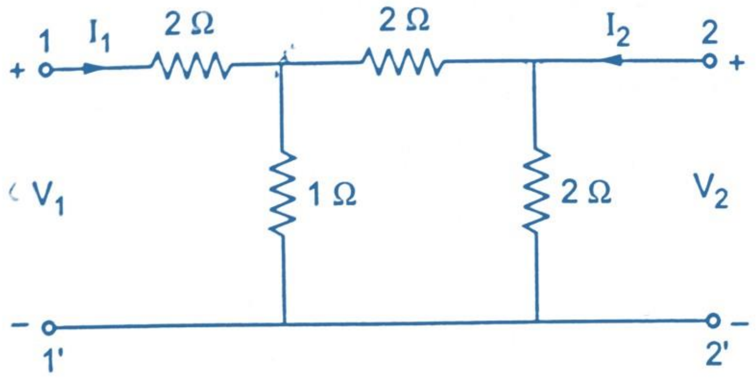
$$z_{12} = z_{21}$$

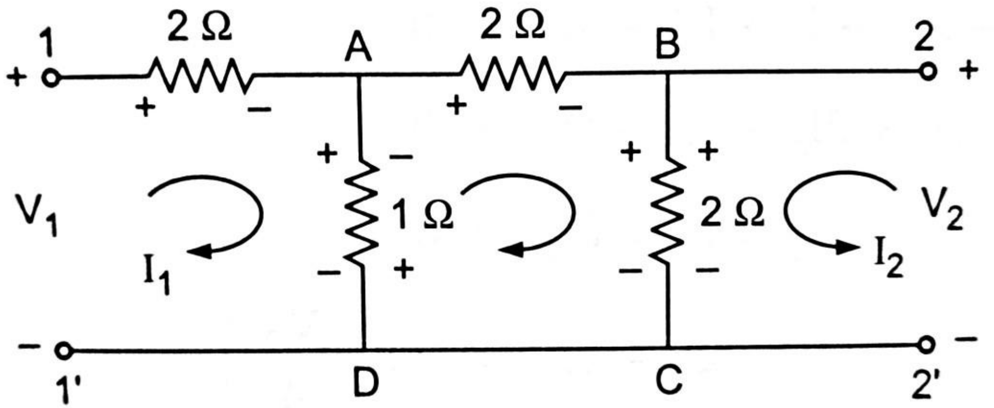
Note:

A **network** is **symmetrical** if its input impedance is equal to its output impedance.

A **network** is said to be **reciprocal** if the voltage appearing at **port 2** due to a current applied at **port** 1is the same as the voltage appearing at **port** 1 when the same current is applied to **port 2**.

Problem1: Determine the z-parameters for the circuit shown.





Applying KVL to the loops,

$-2I_1\ -1I_1\ +1I_3\ +V_1 = 0$

$V_1 = 3I_1 - I_3$ ---------------(1)

$-2I_2\ -2I_3\ +V_2 = 0$

$V_2 = 2I_2 + 2I_3$ -------------(2)

$-2I_3 - 2I_3\ -2I_2\ -1I_3\ +1I_1 = 0$

$5I_3 = I_1 - 2I_2$

$I_3 = (1/5)\ I_1 + (-2/5)\ I_2$ ---------(3)

Substituting value of I_3 in eqn (1)

$V_1 = 3I_1 - [\ 1/5\ I_1 - 2/5\ I_2\ ]$

$V_1 = (14/5)\ I_1 + (2/5)\ I_2$

Substituting value of I_3 in eqn (2)

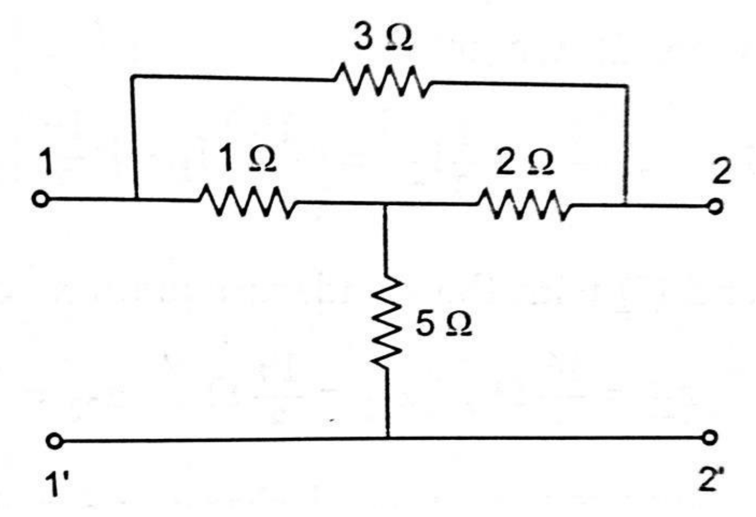
$V_2 = 2I_2 + 2[\ (1/5)\ I_1 + (-2/5)\ I_2]$

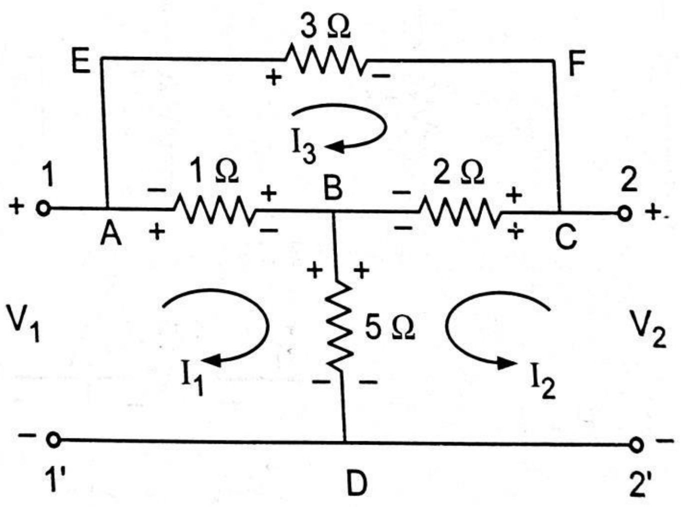
$V_2 = (2/5)\ I_1 + (6/5)\ I_2$

$Z_{11} = 14/5\ \Omega$ $\quad Z_{12} = 2/5\ \Omega$

$Z_{21} = 2/5\ \Omega$ $\quad Z_{22} = 6/5\ \Omega$

Problem2: Determine the z-parameters for the circuit shown.





Applying KVL to the loops,

$-I_1 + 1I_3 - 5I_1 - 5I_2 + V_1 = 0$

$V_1 = 6I_1 + 5I_2 - I_3$ --------------(1)

$-2I_2 - 2I_3 - 5I_2 - 5I_1 + V_2 = 0$

$V_2 = 5I_1 + 7I_2 + 2I_3$ -------------(2)

$-3I_3 - 2I_3 - 2I_2 - 1I_3 + 1I_1 = 0$

$6I_3 = I_1 - 2I_2$

$I_3 = (1/6)\ I_1 + (-1/3)\ I_2$ ---------(3)

Substituting value of I_3 in eqn (1)

$V_1 = 6I_1 + 5I_2 - [\ (1/6)\ I_1 + (-1/3)\ I_2]$

$V_1 = (35/6)\ I_1 + (16/3)\ I_2$

Substituting value of I_3 in eqn (2)

$V_2 = 5I_1 + 7I_2 + 2\ [(1/6)\ I_1 + (-1/3)\ \ ]$

$V_2 = (16/3)\ I_1 + (19/3)\ I_2$

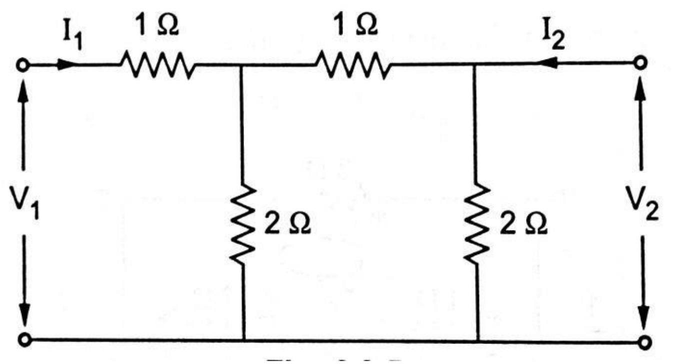
$Z_{11} = 35/6\ \Omega$

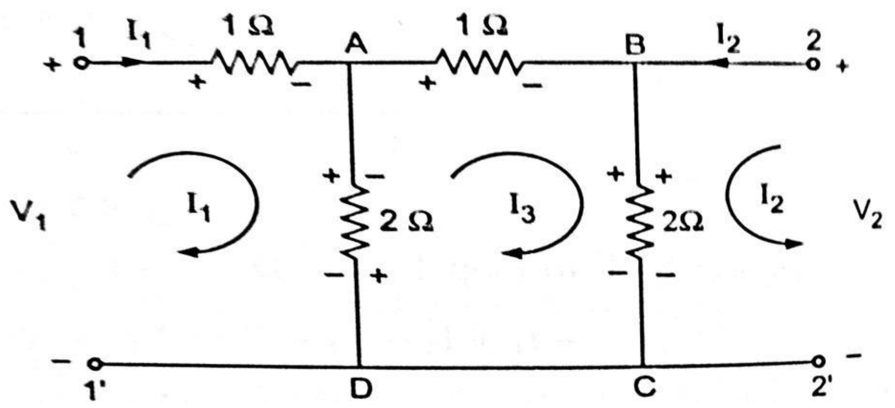
$Z_{12} = 16/3\ \Omega$

$Z_{21} = 16/3\ \Omega$

$Z_{22} = 19/3\ \Omega$

Problem3: Determine the z-parameters for the circuit shown.





Applying KVL to the loops,

$-I_1 - 2I_1 + 2I_3 + V_1 = 0$

$V_1 = 3I_1 - 2I_3$ --------------(1)

$-2I_2 - 2I_3 + V_2 = 0$

$V_2 = 2I_2 + 2I_3$ -------------(2)

$-I_3 - 2I_3 - 2I_2 - 2I_3 + 2I_1 = 0$

$5I_3 = 2I_1 - 2I_2$

$I_3 = (2/5)\, I_1 + (-2/5)\, I_2$ ---------(3)

Substituting value of I_3 in eqn (1)

$V_1 = 3I_1 - 2\,[\,(2/5)\, I_1 + (-2/5)\, I_2]$

$V_1 = (11/5)\, I_1 + (4/5)\, I_2$

Substituting value of I_3 in eqn (2)

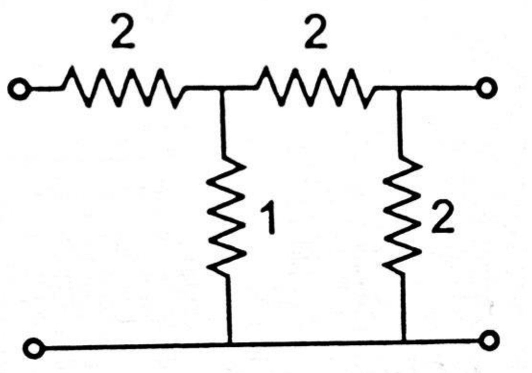
$V_2 = 2I_2 + 2\,[\,(2/5)\, I_1 + (-2/5)\, I_2\,]$

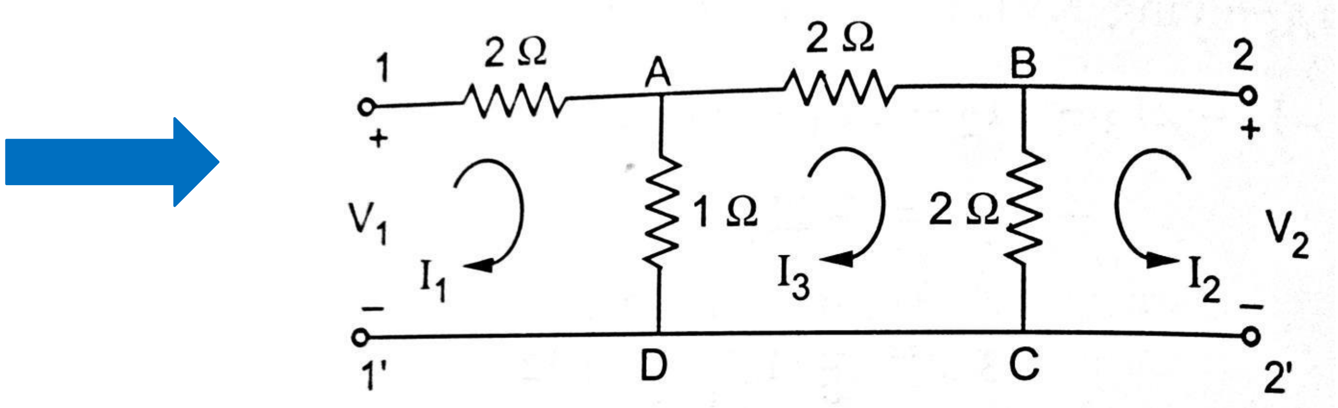
$V_2 = (4/5)\, I_1 + (6/5)\, I_2$

$Z_{11} = 11/5\ \Omega$ $Z_{12} = 4/5\ \Omega$

$Z_{21} = 4/5\ \Omega$ $Z_{22} = 6/5\ \Omega$

Problem4: Determine the z-parameters and also draw the z-parameter equivalent circuit.



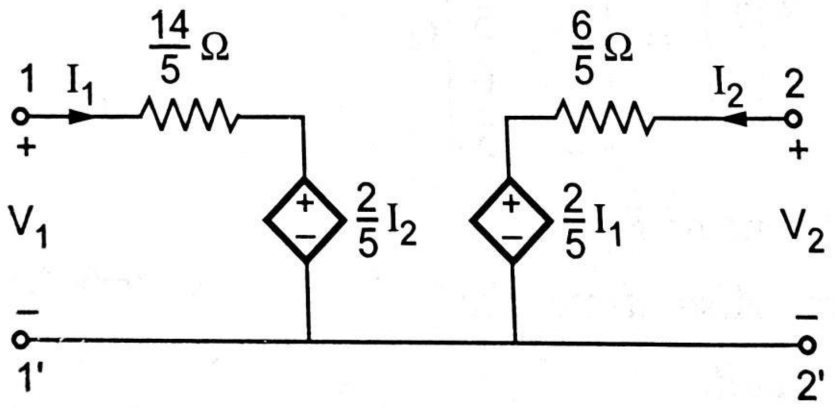


Z_{11}= 14/5 Ω

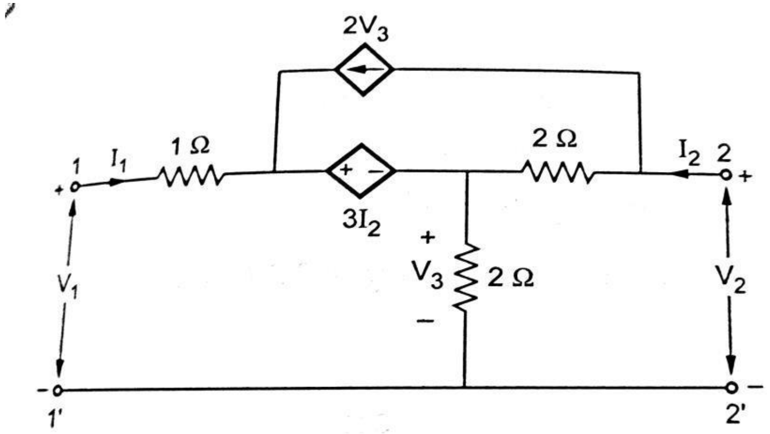
Z_{21}= 2/5 Ω

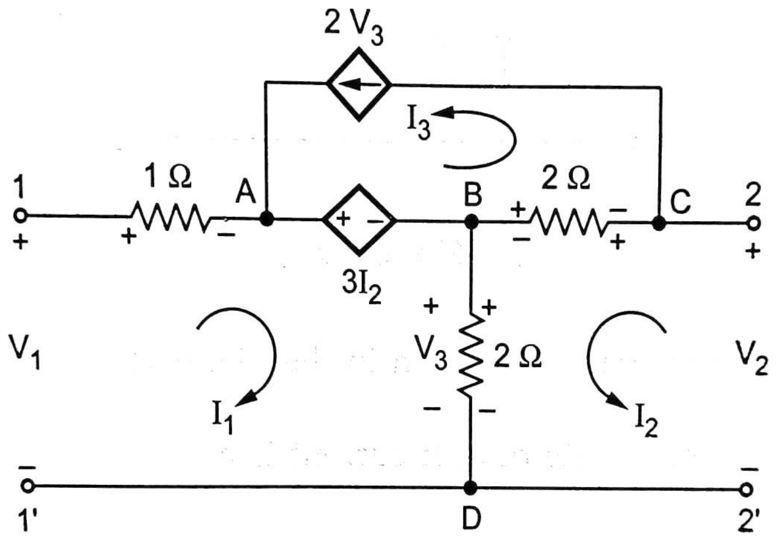
Z_{12}= 2/5 Ω

Z_{22}= 6/5 Ω



Problem5: Determine the z-parameters of the circuit shown.





$V_3 = 2I_1 + 2I_2$

$I_3 = 2V_3 = 4I_1 + 4I_2$

Applying KVL to the loops,

$-I_1 - 3I_2 - 2I_1 - 2I_2 + V_1 = 0$

$V_1 = 3I_1 + 5I_2$ --------------(1)

$-2I_2 + 2I_3 - 2I_2 - 2I_1 + V_2 = 0$

$V_2 = 2I_1 + 4I_2 - 2I_3$

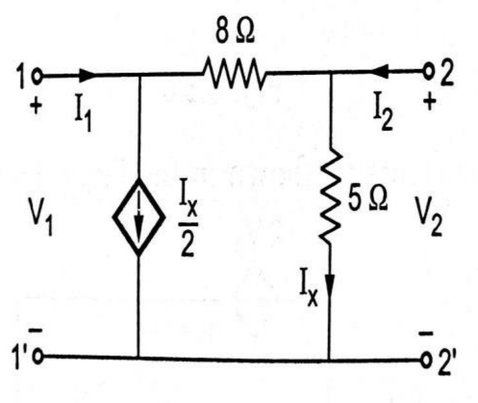
$V_2 = 2I_1 + 4I_2 - 2[\ 4I_1 + 4I_2\ ]$

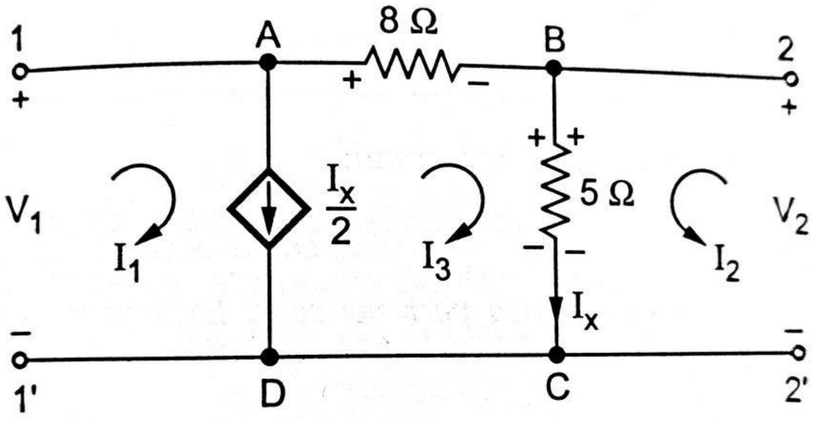
$V_2 = -6I_1 - 4I_2$ ---------------(2)

$Z_{11} = 3\ \Omega$ $Z_{12} = 5\ \Omega$

$Z_{21} = -6\ \Omega$ $Z_{22} = -4\ \Omega$

Problem6: Determine the z-parameters of the circuit shown.





$\mathrm{I_x = I_2 + I_3}$

$I_1 - I_3 = \frac{I_x}{2} = \frac{I_2+I_3}{2}$

$2I_1 - 2I_3 = I_2 + I_3$

$I_3 = 2/3I_1 - 1/3I_2$ ----------(1)

Applying KVL to the loops,

$-8I_3 - 5I_3 - 5I_2 + V_1 = 0$

$V_1 = 13I_3 + 5I_2$ --------------(2)

$-5I_2 - 5I_3 + V_2 = 0$

$V_2 = 5I_2 + 5I_3$ ---------------(3)

Substituting value of I_3 in eqn (2)

$V_1 = 5I_2 + 13\,[2/3I_1 - 1/3I_2]$

$V_1 = (26/3)\,I_1 + (2/3)\,I_2$

Substituting value of I_3 in eqn (3)

$V_2 = 5I_2 + 5\,[2/3I_1 - 1/3\ ]$

$V_2 = (10/3)\,I_1 + (10/3)\,I_2$

$Z_{11} = 26/3\ \Omega$ $\quad Z_{12} = 2/3\ \Omega$

$Z_{21} = 10/3\ \Omega$ $\quad Z_{22} = 10/3\ \Omega$

y Parameters or Short Circuit Admittance Parameters

- They are obtained by expressing currents at two ports in terms of voltage at two ports.
- $I_1 and\ I_2$ are dependent variables
- $V_1 and\ V_2$ are independent variables

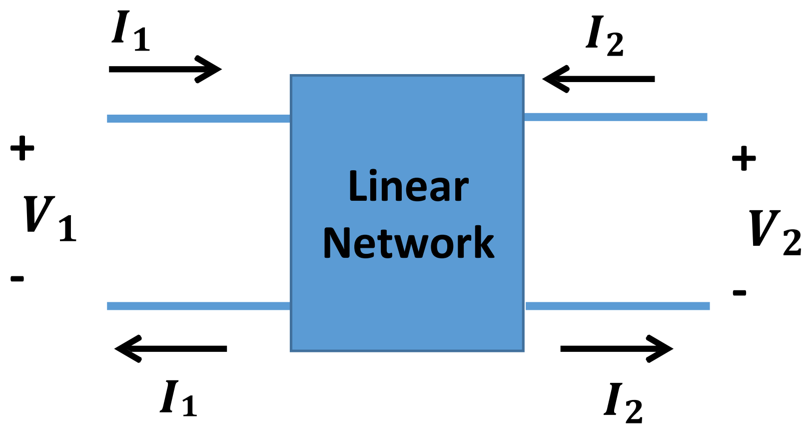
$$I_1 = f_1\,(V_1\,, V_2\,)$$

$$I_2 = f_2\,(V_1\,, V_2\,)$$

Above equations and be written as

$$I_1 = y_{11}\,V_1 + y_{12}V_2 \quad .....eq\,(1)$$

$$I_2 = y_{21}\,V_1 + y_{22}V_2 \quad .....eq\,(2)$$



In matrix form, equations can be written as

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$I = y\ V$$

y Parameters or Short Circuit Admittance Parameters

- The individual y Parameters can be obtained as follows

$$I_1 = y_{11} V_1 + y_{12} V_2$$

$$I_2 = y_{21} V_1 + y_{22} V_2$$

1. To obtain y_{11}

$$Let\ V_2 = 0 \rightarrow Port\ 2\ is\ short\ circuited$$

From eq (1) $\quad I_1 = y_{11} V_1$

$$\boxed{y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} \ \mho}$$

The parameter y_{11} is called short circuit driving point input admittance.

2. To obtain y_{21}

$$Let\ V_2 = 0 \rightarrow Port\ 2\ is\ short\ circuited$$

From eq (2) $\quad I_2 = y_{21} V_1$

$$\boxed{y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} \ \mho}$$

The parameter y_{21} is called short circuit forward transfer admittance.

y Parameters or Short Circuit Admittance Parameters

$$I_1 = y_{11} V_1 + y_{12} V_2$$

$$I_2 = y_{21} V_1 + y_{22} V_2$$

3. To obtain y_{12}

$Let\ V_1 = 0 \rightarrow Port\ 1\ is\ short\ circuited$

From eq (1) $\quad I_1 = y_{12} V_2$

$$\boxed{y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} \ \mho}$$

The parameter y_{12} is called short circuit reverse transfer admittance.

4. To obtain y_{22}

$Let\ V_1 = 0 \rightarrow Port\ 1\ is\ short\ circuited$

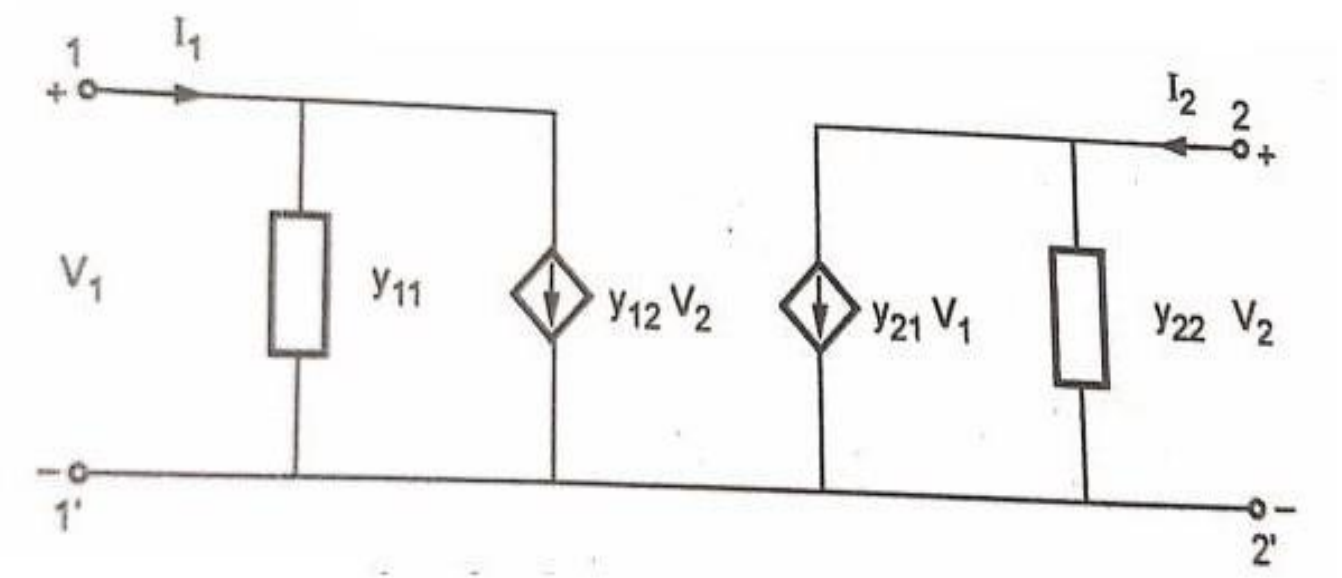
From eq (2) $\quad I_2 = y_{22} V_2$

$$\boxed{y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} \ \mho}$$

The parameter y_{22} is called short circuit driving point output admittance.

y Parameters or Short Circuit Admittance Parameters

➢ **Equivalent Network in terms of y parameters**



➢ **Conditions for Symmetry**

$$y_{11} = y_{22}$$

➢ **Conditions for Reciprocity**

$$y_{12} = y_{21}$$

Problem 1: Following short circuit currents and voltages are obtained experimentally for a two port network.

i) With output port short circuited: $I_1 = 5mA, I_2 = -0.3mA, V_1 = 25V$

ii) With input port short circuited: $= -5mA, I_2 = 10\ mA, V_2 = 30V$

Determine y parameters.

i) Output port short circuited → $V_2 = 0V$

By definition

$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{5 * 10^{-3}}{25} = 0.2\ \text{m}\ \mho$$

$$y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \frac{-0.3 * 10^{-3}}{25} = -0.012\ \text{m}\ \mho$$

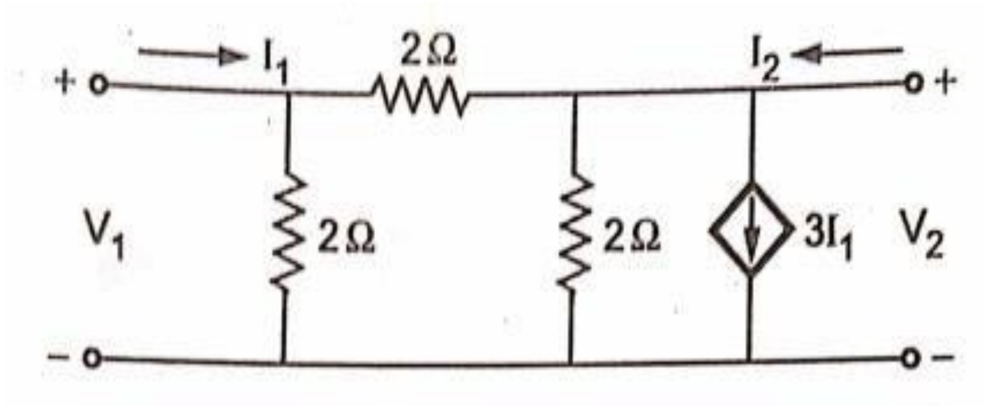
ii) Input port short circuited → $V_1 = 0V$

By definition

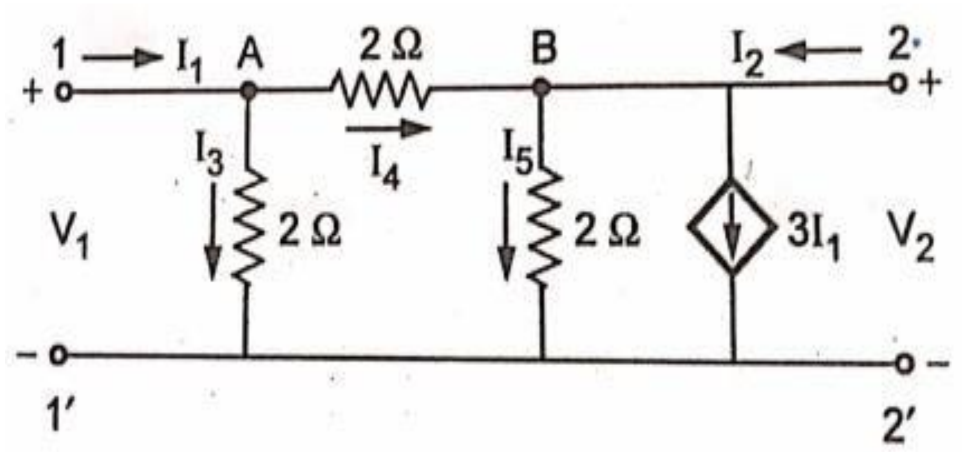
$$y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \frac{-5 * 10^{-3}}{30} = -0.16667\text{m}\ \mho$$

$$y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{10 * 10^{-3}}{30} = 0.3333\ \text{m}\ \mho$$

Problem2: Determine the y-parameters for the two port circuit shown in figure.



Assume two nodes A and B and also assume branch currents



y-parameters are given by

$I_1 = y_{11} V_1 + y_{12} V_2$

$I_2 = y_{21} V_1 + y_{22} V_2$

From fig

$V_A = V_1$

$V_B = V_2$

Apply KCL at node A

$I_1 - I_3 - I_4 = 0$

$I_1 = I_3 + I_4 = \frac{V_1}{2} + \frac{V_1 - V_2}{2}$

$I_1 = V_1 - 0.5V_2$ ... (1)

Apply KCL at node B

$I_4 - I_5 - 3I_1 + I_2 = 0$

$I_2 = 3I_1 + I_5 - I_4$

$I_2 = 3I_1 + \frac{V_2}{2} - \frac{V_1 - V_2}{2}$ ... (2)

Substitute eq(1) in eq(2)

$I_2 = 3(V_1 - 0.5V_2) + 0.5V_2 - 0.5V_1 + 0.5V_2$

$I_2 = 2.5V_1 - 0.5V_2$ .... (3)

Comparing eq(1) and eq(3) we get y parameters

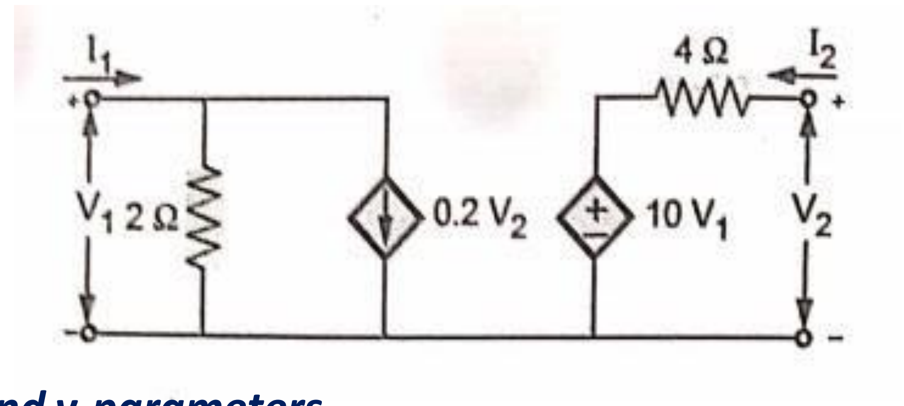
$y_{11} = 1$ u

$y_{12} = -0.5$ u

$y_{21} = 2.5$ u

$y_{22} = -0.5$ u

Problem3: Using the definitions, find the y-parameters of the two port network shown in figure.



y-parameters are given by

$$I_1 = y_{11} V_1 + y_{12} V_2$$
$$I_2 = y_{21} V_1 + y_{22} V_2$$

To find y-parameters

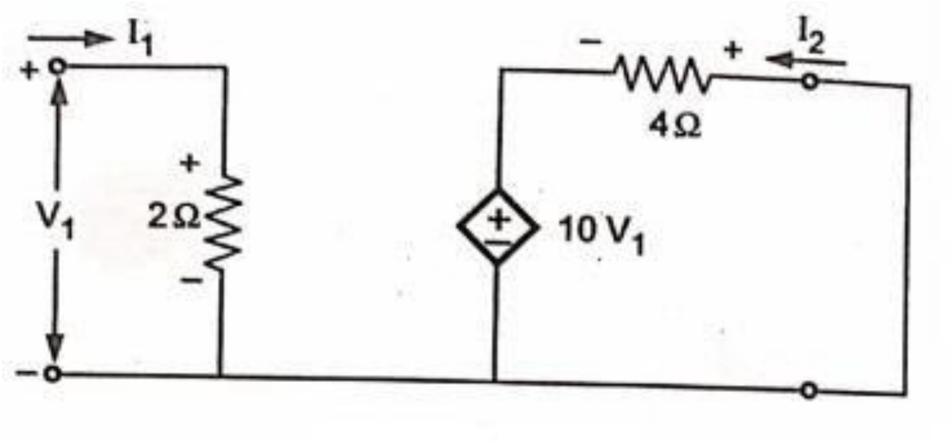
1. $Let\ V_2 = 0 \rightarrow Port\ 2\ is\ short\ circuited$

Since $V_2 = 0$,

$the\ value\ of\ dependent\ source\ 0.2V_2 = 0$

$$From\ fig{:}\ \ V_1 = 2I_1$$

$$\therefore y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{1}{2} = 0.5\ ℧$$



$Apply\ KVL\ at\ output\ side$

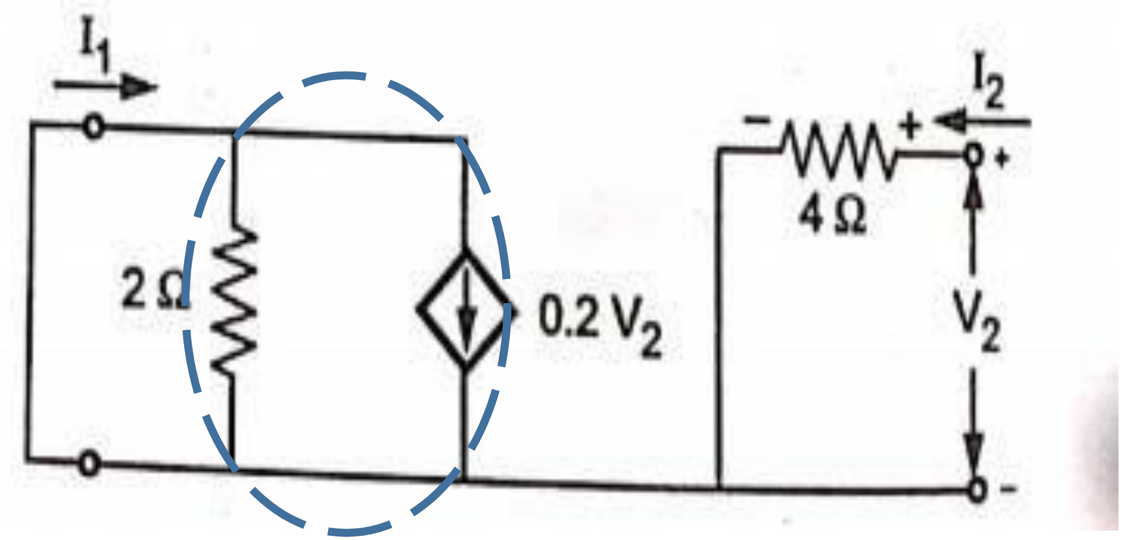
$$4I_2 + 10V_1 = 0$$
$$4I_2 = -10V_1$$

$$\therefore y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -2.5\ ℧$$

y-parameters are given by

$I_1 = y_{11} V_1 + y_{12} V_2$

$I_2 = y_{21} V_1 + y_{22} V_2$



To find y-parameters

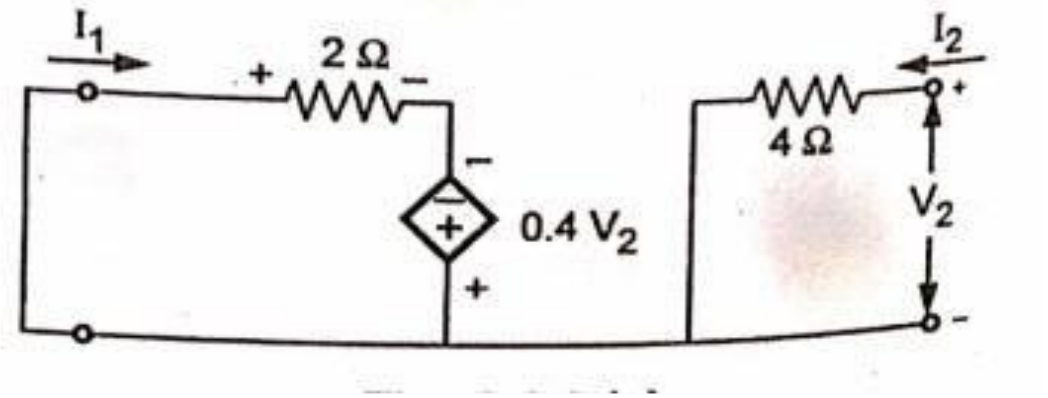
$1.\ Let\ V_1 = 0 \rightarrow Port\ 1\ is\ short\ circuited$

Since $V_1 = 0$,

$the\ value\ of\ dependent\ source\ 10V_1 = 0$

$From\ fig\!:\ V_2 = 4I_2$

$\therefore\ y_{22} = \left.\frac{I_2}{V_2}\right|_{V_2=0} = \frac{1}{4} = 0.25\ \mho$



Convert Current source to voltage source

Apply KVL at output side

$-2I_1 + 0.4V_2 = 0$

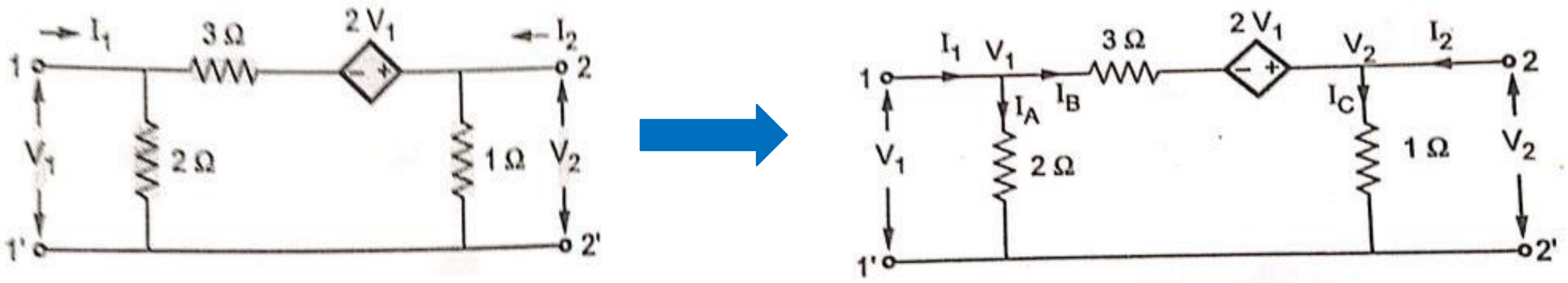
$0.4V_2 = 2I_1$

$\therefore\ y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = 0.2\ \mho$

The y-parameters are

$$[y] = \begin{bmatrix} 0.5 & 0.2 \\ -2.5 & 0.25 \end{bmatrix}$$

Problem4: **Determine y-parameters for the network shown in figure.**



y-parameters are given by

$I_1 = y_{11} V_1 + y_{12} V_2$

$I_2 = y_{21} V_1 + y_{22} V_2$

Apply KCL at node 1

$$I_1 - I_A - I_B = 0$$

$$I_1 = I_A + I_B$$

$$I_1 = \frac{V_1 - 0}{2} + \left(\frac{(V_1 + 2V_1) - V_2}{3}\right)$$

$$I_1 = \frac{3V_1}{2} - \frac{V_2}{3} \quad \ldots. (1)$$

Apply KCL at node 2

$$I_B + I_2 - I_C = 0$$

$$I_2 = I_C - I_B$$

$$I_2 = \frac{V_2 - 0}{1} - \left(\frac{(V_1 + 2V_1) - V_2}{3}\right)$$

$$I_2 = -V_1 + \frac{4V_2}{3} \quad \ldots. (2)$$

Comparing eq(1) and eq(2) we get y parameters

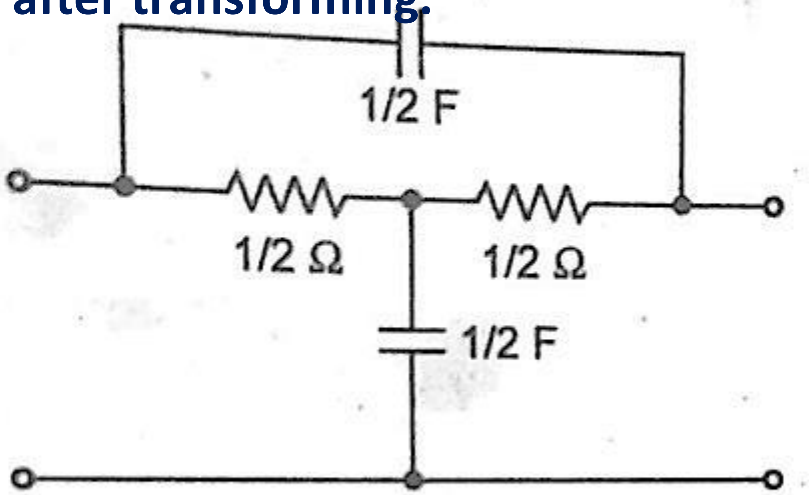
$$y_{11} = \frac{3}{2}\ \text{u}$$

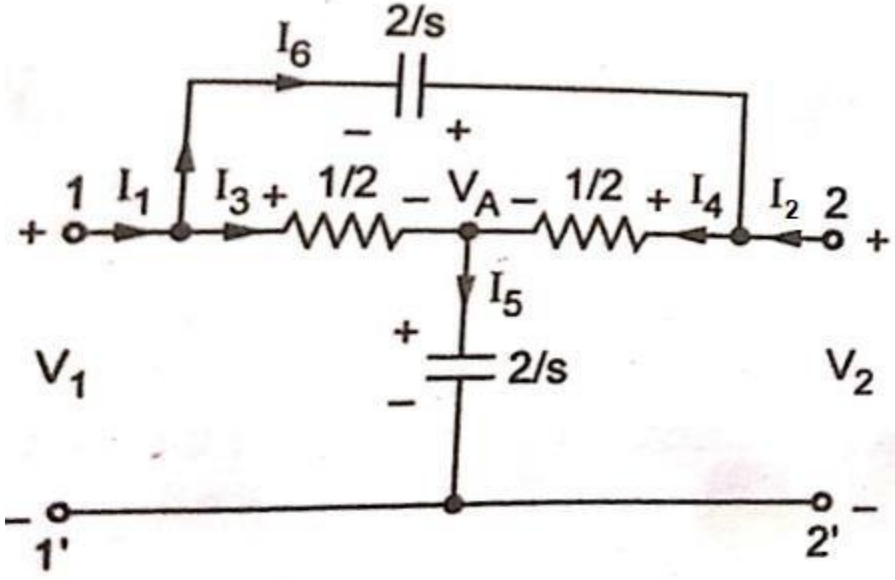
$$y_{12} = -\frac{1}{3}\ \text{u}$$

$$y_{21} = -1\text{u}$$

$$y_{22} = \frac{4}{3}\ \text{u}$$

Problem 5: **The bridged T-RC network is shown in figure. For the values given, find the y – parameters after transforming.**





Apply KCL at node 1

$$I_1 - I_3 - I_6 = 0$$

$$I_1 = I_3 + I_6$$

$$I_1 = \frac{V_1 - V_A}{1/2} + \frac{V_1 - V_2}{2/S}$$

$$I_1 = 2V_1 - 2V_A + \frac{S}{2}V_1 - \frac{S}{2}V_2$$

$$I_1 = \left(\frac{S+4}{2}\right)V_1 - \frac{S}{2}V_2 - 2V_A \ldots (1)$$

Apply KCL at node 2

$$-I_2 + I_4 - \quad = 0$$

$$I_2 = I_4 - I_6$$

$$I_2 = \frac{V_2 - V_A}{1/2} - \frac{V_1 - V_2}{2/S}$$

$$I_2 = 2V_2 - 2V_A - \frac{S}{2}V_1 + \frac{S}{2}V_2$$

$$I_2 = -\frac{S}{2}V_1 + \left(\frac{S+4}{2}\right)V_2 - 2V_A \ldots (2)$$

Apply KCL at node A

$$I_3 + I_4 - I_5 = 0$$

$$I_3 + I_4 = I_5$$

$$\frac{V_1 - V_A}{1/2} + \frac{V_2 - V_A}{1/2} = \frac{V_A - 0}{2/S}$$

$$2V_1 - 2V_A + 2V_2 - 2V_A = \frac{S}{2}V_A$$

$$\left(\frac{S}{2} + 4\right)V_A = 2V_1 + 2V_2$$

$$\left(\frac{S+8}{2}\right)V_A = 2V_1 + 2V_2$$

$$V_A = \left(\frac{4}{S+8}\right)V_1 + \left(\frac{4}{S+8}\right)V_2 \quad \ldots(3)$$

Substitute eq (3)in eq (1)

$$I_1 = \left(\frac{S+4}{2}\right)V_1 + \frac{S}{2}V_2 - 2\left(\left(\frac{4}{S+8}\right)V_1 + \left(\frac{4}{S+8}\right)V_2\right)$$

$$I_1 = \left(\frac{S+4}{2} - \frac{8}{S+8}\right)V_1 - \left(\frac{S}{2} + \frac{8}{S+8}\right)V_2$$

$$I_1 = \left(\frac{s^2+12S+32-16}{2(s+8)}\right)V_1 - \left(\frac{s^2+8S++16}{2(s+8)}\right)V_2$$

$$I_1 = \left(\frac{s^2+12S+16}{2(s+8)}\right)V_1 - \left(\frac{s^2+8S+16}{2(s+8)}\right)V_2 \quad \ldots(4)$$

Substitute eq (3)in eq (2)

$$I_2 = -\frac{S}{2}V_1 + \left(\frac{S+4}{2}\right)V_2 - 2\left(\left(\frac{4}{S+8}\right)V_1 + \left(\frac{4}{S+8}\right)V_2\right)$$

$$I_2 = \left(-\frac{S}{2} - \frac{8}{S+8}\right)V_1 + \left(\frac{S+4}{2} - \frac{8}{S+8}\right)V_2$$

$$I_2 = -\left(\frac{s^2+8S+16}{2(s+8)}\right)V_1 + \left(\frac{s^2+12S+16}{2(s+8)}\right)V_2 \quad \ldots(5)$$

y parameters are given by (eq(4) and eq(5))

$$[y] = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{s^2 + 12S + 16}{2(\, s + 8)}\right) & \left(\frac{s^2 + 8S + +16}{2(\, s + 8)}\right) \\ -\left(\frac{s^2 + 8S + 16}{2(\, s + 8)}\right) & \left(\frac{s^2 + 12S + +16}{2(\, s + 8)}\right) \end{bmatrix}$$

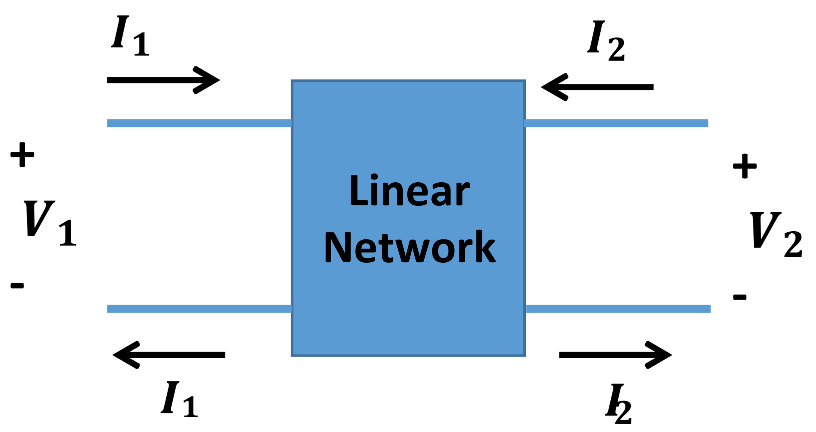
h Parameters or Hybrid Parameters

- Useful for constructing models for transistors.
- They are obtained by expressing voltages at the input port and current at the output port in terms of current at the input port and voltage at the output port.
- I and V_2 are independent variables
- I and V_1 are dependent variables

$$V_1 = f_1\,(I_1\,, V_2\,)$$

$$I_2 = f_2\,(I_1\,, V_2\,)$$

Above equations and be written as



$$V_1 = h_{11}\,I_1 + h_{12}V_2 \quad \ldots\ldots eq\ (1)$$

$$I_2 = h_{21}\,I_1 + h_{22}V_2 \quad \ldots\ldots eq\ (2)$$

In matrix form, equations can be written as

$$\begin{matrix} V_1 \\ I_2 \end{matrix} = \begin{matrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{matrix} \begin{matrix} I_1 \\ V_2 \end{matrix}$$

h Parameters or Hybrid Parameters

- The individual h Parameters can be obtained as follows

$$V_1 = h_{11} I_1 + h_{12} V_2$$
$$I_2 = h_{21} I_1 + h_{22} V_2$$

1. To obtain h_{11}

$$Let\ V_2 = 0 \rightarrow Port\ 2\ is\ short\ circuited$$

From eq (1) $V_1 = h_{11} I_1$

$$\boxed{h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}\ \Omega}$$

The parameter h_{11} is called short circuit input impedance.

2. To obtain h_{21}

$$Let\ V_2 = 0 \rightarrow Port\ 2\ is\ short\ circuited$$

From eq (2) $I_2 = h_{21} I_1$

$$\boxed{h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}}$$

The parameter h_{21} is called short circuit forward current gain. It is unitless.

h Parameters or Hybrid Parameters

$$V_1 = h_{11} I_1 + h_{12} V_2$$
$$I_2 = h_{21} I_1 + h_{22} V_2$$

3. To obtain h_{12}

$Let\ I_1 = 0 \rightarrow Port\ 1\ is\ open\ circuited$

From eq (1) $V_1 = h_{12} V_2$

$$\boxed{h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}}$$

The parameter h_{12} is called open circuit reverse voltage gain. It is unitless.

4. To obtain h_{22}

$Let\ I_1 = 0 \rightarrow Port\ 1\ is\ open\ circuited$

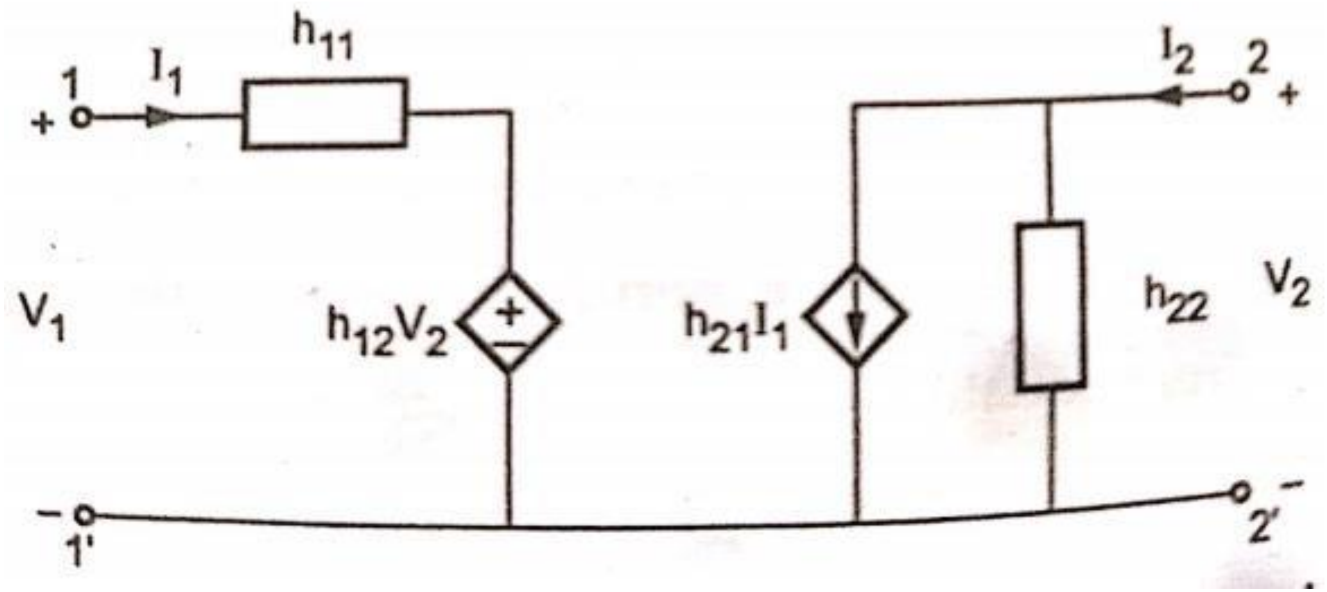
From eq (2) $I_2 = h_{22} V_2$

$$\boxed{h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} \ \mho}$$

The parameter h_{22} is called open circuit output admittance.

h Parameters or Hybrid Parameters

- Equivalent Network in terms of h parameters



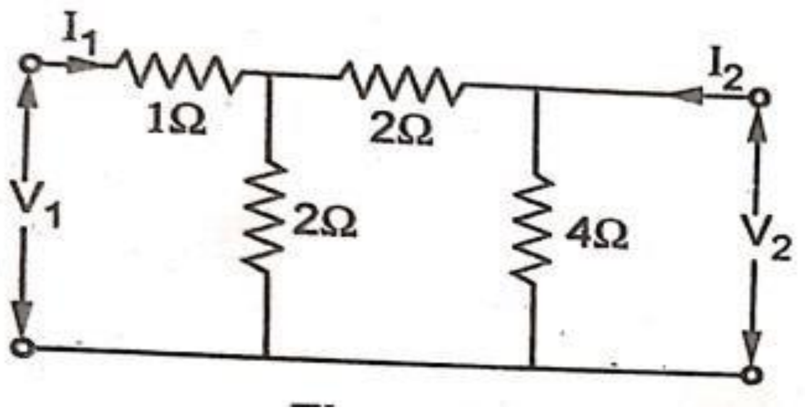
- Conditions for Symmetry

$$h_{11}\, h_{22} - h_{12}\, h_{21} = 1$$

- Conditions for Reciprocity

$$h_{12} = -h_{21}$$

Problem1: **Find the h-parameters of the network shown in figure . Give its equivalent Circuit.**



h-parameters are given by

$V_1 = h_{11} I_1 + h_{12} V_2$

$I_2 = h_{21} I_1 + h_{22} V_2$

To find h-parameters

1. Let $= 0 \rightarrow$ ***Port 2 is short circuited***

Since $V_2 = 0$, 4Ω ***resistor is short circuited.***

Apply KCL at node (a)

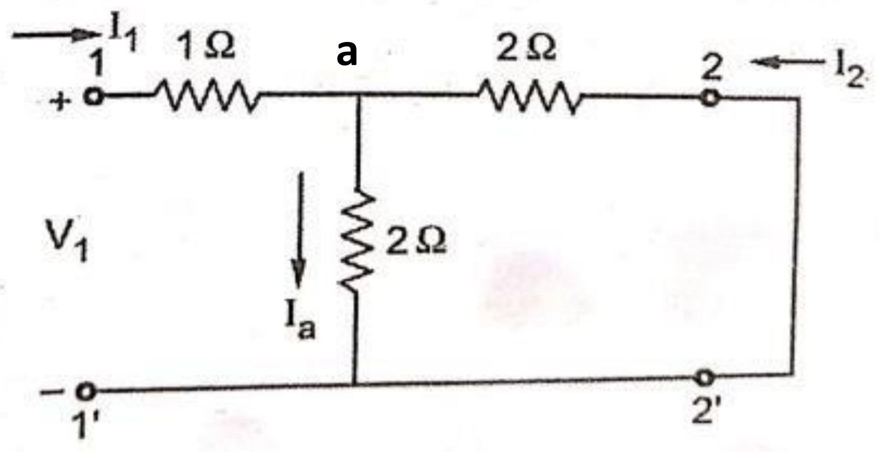
$$I_a = I_1 + I_2$$

By current divider rule

$$I_2 = -I_1 \left[\frac{2}{2+2}\right] = \frac{-I_1}{2}$$

$$\frac{I_2}{I_1} = \frac{-1}{2} \quad \text{....} (1)$$

Hence $h_{21} = \frac{I_2}{I_1}\Big|_{V_2=0} = \frac{-1}{2}$



Apply KVL at the input side

$$V_1 = I_1 + I_a(2) = I_1 + 2(I_1 + I_2)$$

Subsitiute for I_2 from eq(1)

$$V_1 = 3I_1 + 2I_2 = 3I_1 + 2(\frac{-1}{2}I_1)$$

$$V_1 = 2\, I_1$$

Hence

$$h_{11} = \frac{V_1}{I_1}\Big|_{V_2=0} = 2\,\Omega$$

To find h-parameters

2. Let $I_1 = 0 \rightarrow$ Port 1 is open circuited

Current I_b is given by

$$I_b = \frac{V_1}{2}$$

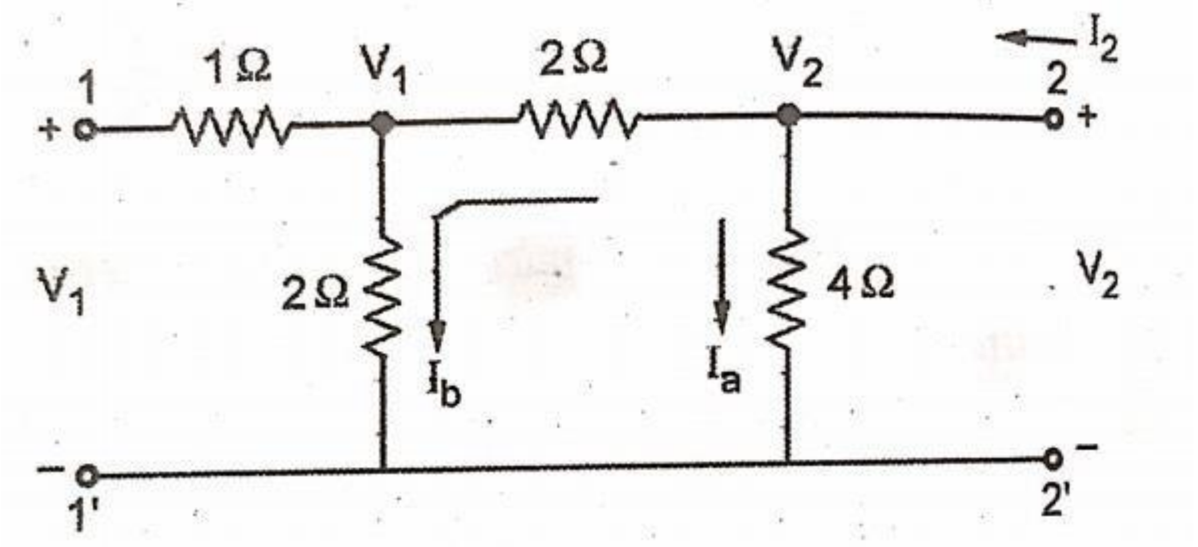
Also $I_b = \frac{V_2 - V_1}{2}$

Equating equations of I_b

$$\frac{V_1}{2} = \frac{V_2 - V_1}{2}$$

$$V_2 = 2\,V_1$$

Hence, $h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{1}{2}$



Apply KCL at the node V_2

$$I_2 = I_a + I_b$$

$$I_2 = \frac{V_2}{4} + \frac{V_2 - V_1}{2}$$

$$I_2 = \frac{V_2}{4} + \frac{V_2 - (\frac{1}{2})V_2}{2}$$

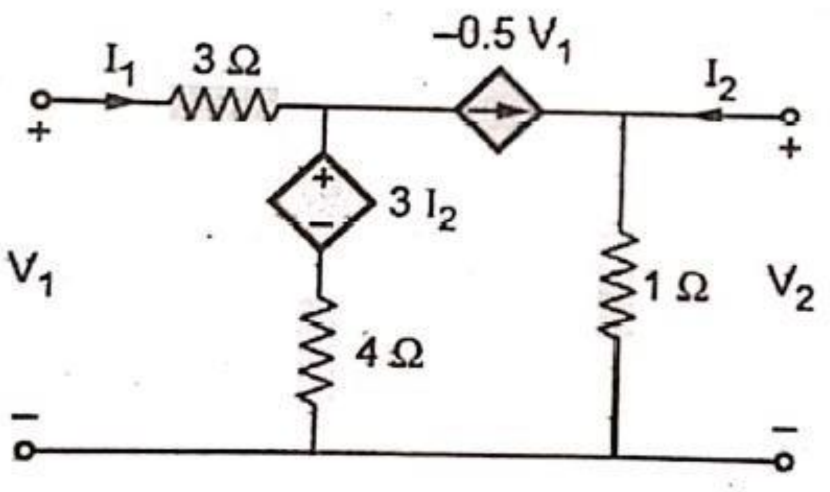
$$I_2 = \frac{V_2}{2}$$

Hence, $h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{1}{2}\,℧$

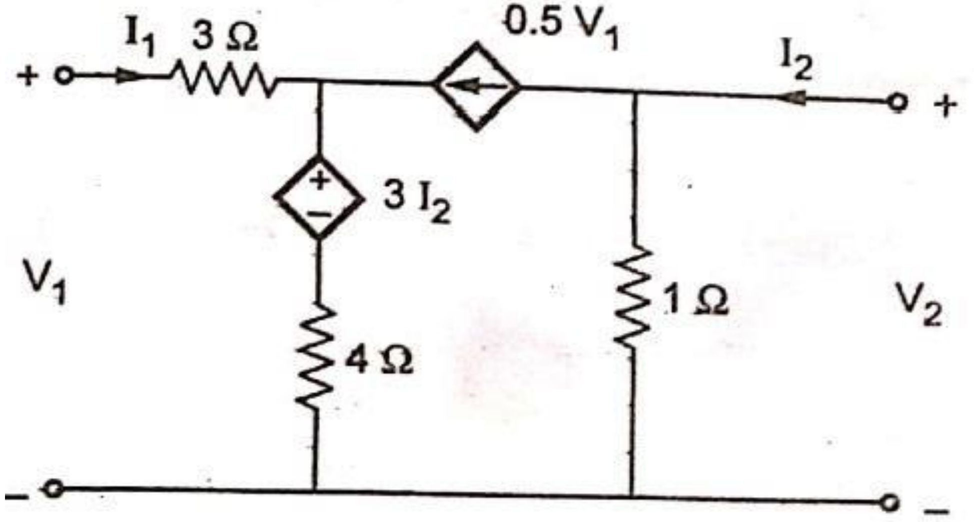
The h-parameters are

$$[h] = \begin{bmatrix} 2 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Problem2: **Find the h-parameters for the two-port network shown in figure .**



Redraw circuit by making positive value of dependent current source by reversing its direction



h-parameters are given by

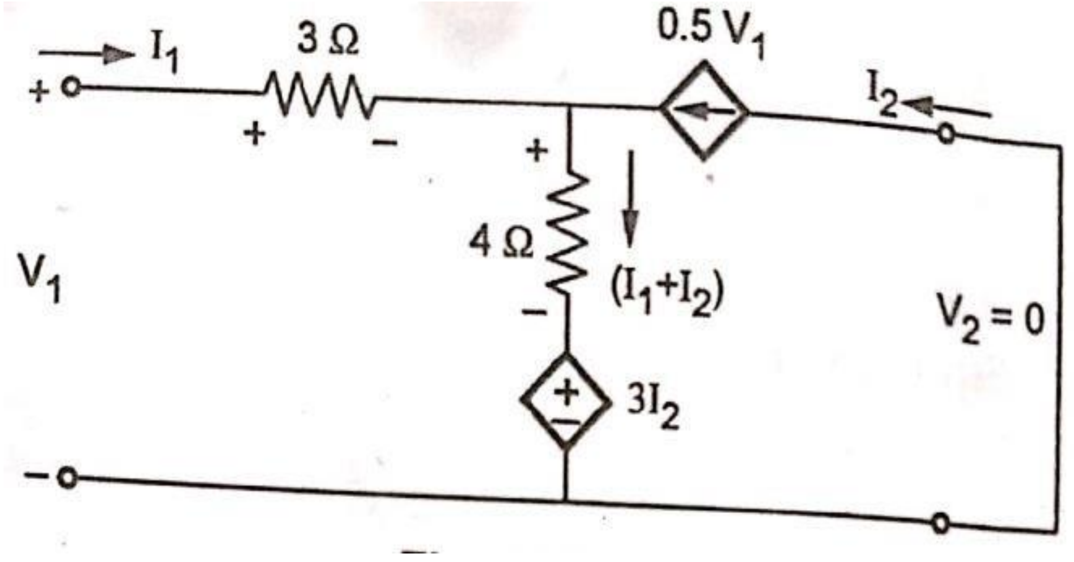
$V_1 = h_{11} I_1 + h_{12} V_2$

$I_2 = h_{21} I_1 + h_{22} V_2$

To find h-parameters

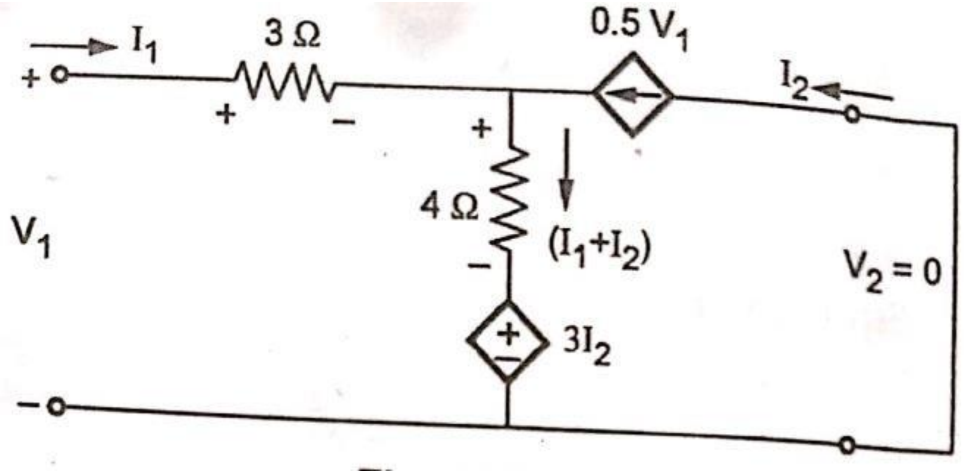
1. Let $= 0 \rightarrow$ ***Port 2 is short circuited***

Since $V_2 = 0$, 1Ω ***resistor is short circuited.***



From Figure

$I_2 = 0.5\,V_1$ .... (1)



Apply KVL to loop 1

$$V_1 - 3I_1 - 4(I_1 + I_2) - 3I_2 = 0$$
$$V_1 = 7I_1 + 7I_2 \quad \dots(2)$$

Substitute eq(1) in eq(2)

$$V_1 = 7I_1 + 7(0.5V_1)$$
$$V_1 = 7I_1 + 3.5V_1$$
$$-2.5V_1 = 7I_1$$

$$\frac{V_1}{I_1} = -2.8$$

$$Hence, h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = -2.8\,\Omega$$

$$V_1 = -2.8I_1 \dots (3)$$

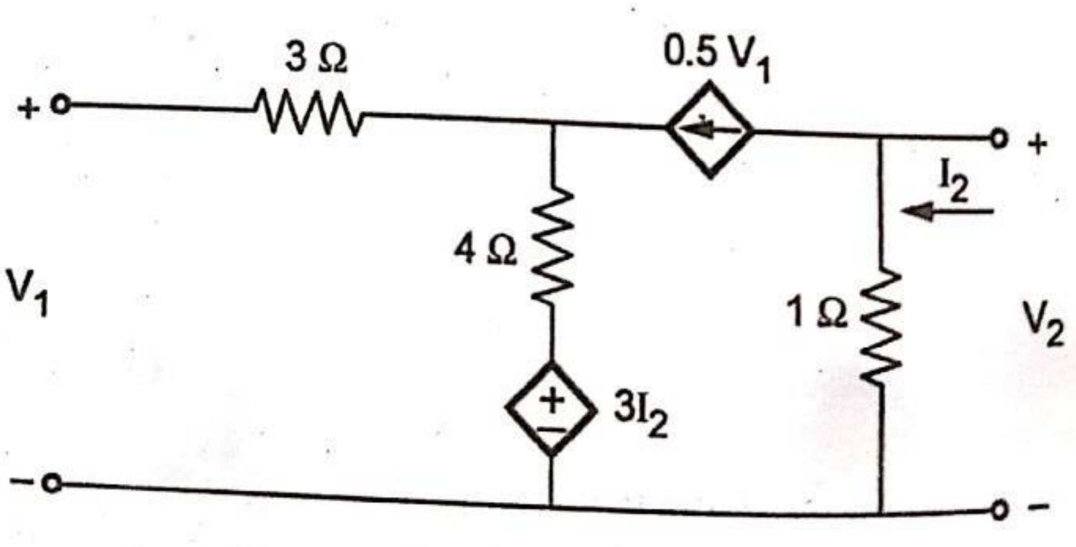
Substitute eq(3) in eq(1)

$$I_2 = 0.5\,V_1 = 0.5(-2.8I_1)$$
$$\frac{I_2}{I_1} = -1.4$$

$$Hence,\ h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = -1.4$$

To find h-parameters

2. Let $I_1 = 0 \rightarrow$ Port 1 is open circuited



Apply KCL to node 2

$$I_2 = \frac{V_2}{1} + 0.5V_1 \quad \dots (4)$$

Apply KVL to loop 1

$$V_1 = 4(0.5V_1) + 3I_2$$
$$V_1 = 2V_1 + 3I_2$$
$$-V_1 = 3I_2 \dots (5)$$

Substituting eq(5) in eq(4)

$$I_2 = \frac{V_2}{1} + 0.5(-3I_2)$$
$$2.5I_2 = V_2$$
$$\frac{I_2}{V_2} = \frac{1}{2.5} = 0.4$$

$$hence,\ h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = 0.4\mho$$

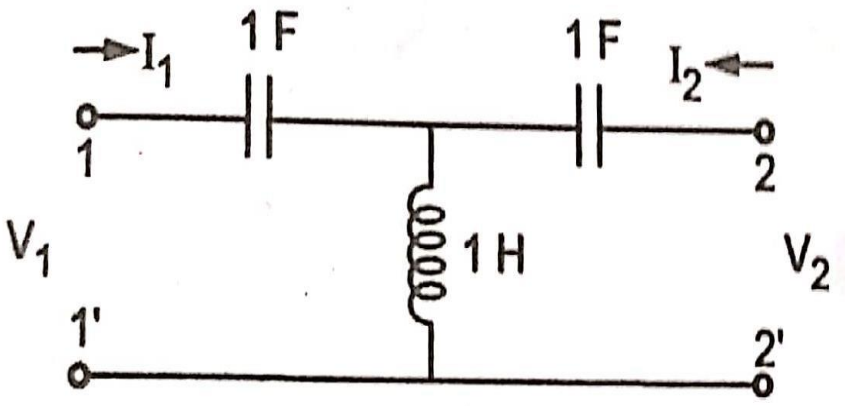
Substituting eq(4) in eq(5)

$$-V_1 = 3((V_2 + 0.5V_1))$$
$$-V_1 = 3V_2 + 1.5V_1$$
$$-2.5V_1 = 3V_2$$

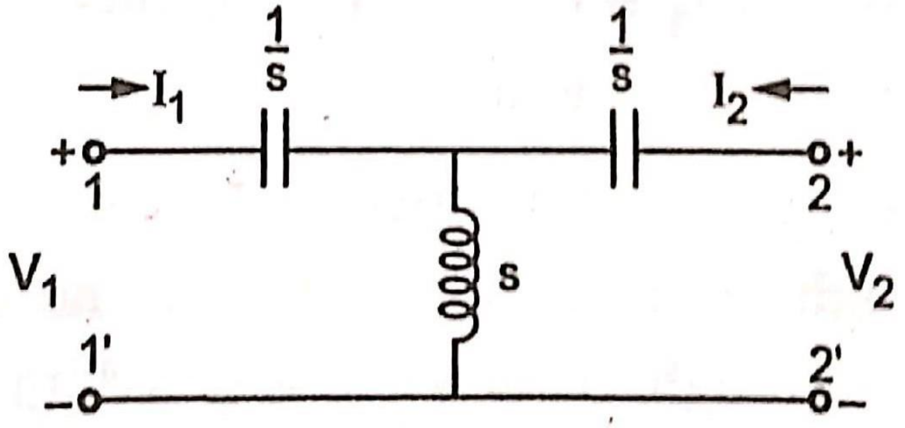
$$\frac{V_1}{V_2} = -1.2$$

$$Hence,\ h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = -1.2$$

Problem3: Find the h-parameters after writing transformed network.



Transformed network



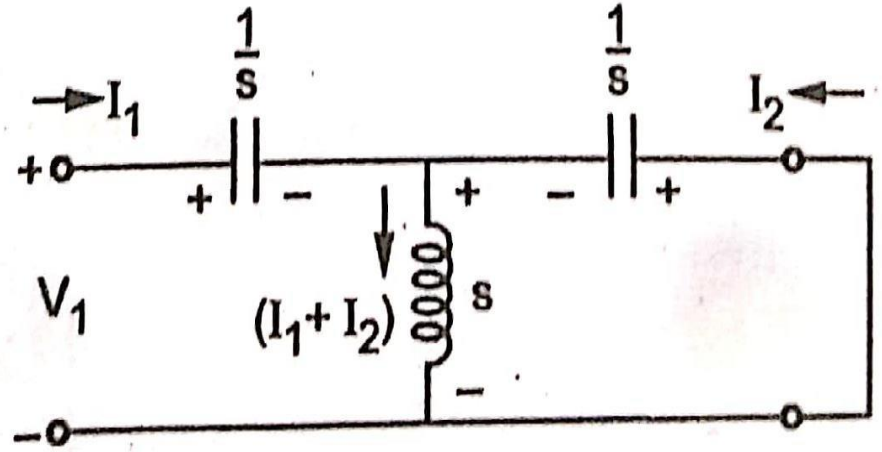
h-parameters are given by

$V_1 = h_{11} I_1 + h_{12} V_2$

$I_2 = h_{21} I_1 + h_{22} V_2$

To find h-parameters

1. $\mathit{Let}\ V_2 = 0 \ \rightarrow \mathit{Port\ 2\ is\ short\ circuited}$



From Figure, apply current divider rule

$$I_2 = -I_1\left(\frac{S}{S+\frac{1}{S}}\right) = -\left(\frac{S^2}{S^2+1}\right) I_1 \quad \ldots (1)$$

Apply KVL at the input side

$$V_1 - \frac{1}{S} I_1 - S(I_1 + I_2) = 0$$

$$V_1 = \left(\frac{1}{S} + S\right) I_1 + S(I_2) \quad \ldots (2)$$

Substitute eq(1) in eq(2)

$$V_1 = \left(\frac{1}{S}+S\right)I_1 + S\left[-\left(\frac{S^2}{S^2+1}\right)I_1\right]$$

$$V_1 = \left[\left(\frac{1+S^2}{S}\right) - \left(\frac{S^3}{S^2+1}\right)\right] I_1$$

$$V_1 = \left[\frac{(1+S^2)(S^2+1) - S^4}{S(S^2+1)}\right] I_1$$

$$V_1 = \frac{1+2S^2+S^4-S^4}{S(S^2+1)} I_1$$

$$\frac{V_1}{I_1} = \frac{2S^2+1}{S(S^2+1)}$$

$$\text{Hence, } h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{2S^2+1}{S(S^2+1)}\,\Omega$$

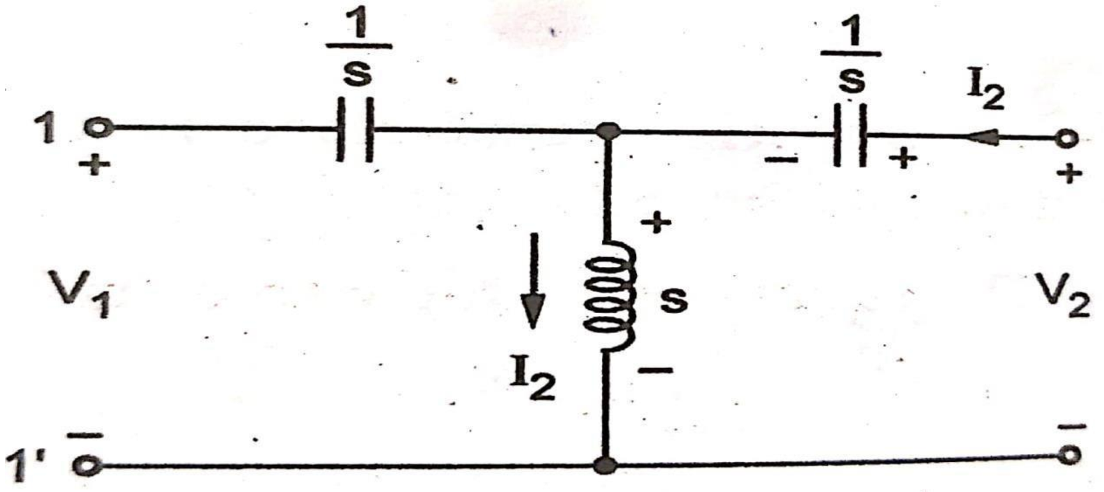
From Eq (1)

$$\frac{I_2}{I_1} = \left(\frac{S^2}{S^2+1}\right)$$

$$\text{Hence, } h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = -\frac{S^2}{S^2+1}$$

To find h-parameters

2. Let $I_1 = 0 \rightarrow$ *Port 1 is open circuited*



Apply KVL at the output side

$$V_2 - \frac{1}{S}I_2 - S\,I_2 = 0$$

$$-\left(\frac{1}{S}+S\right) I_2 = -V_2$$

$$\left(\frac{1+S^2}{S}\right) I_2 = V_2 \rightarrow I_2 = \left(\frac{S}{S^2+1}\right) V_2 \quad \text{-- (3)}$$

$$\frac{I_2}{V_2} = \left(\frac{S}{S^2+1}\right)$$

$$\text{hence, } h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{S}{S^2+1}\,\mho$$

From Figure

$$V_1 = s\,I_2 \text{ --- } (4)$$

Substitute eq (3) in eq (4)

$$V_1 = s\left(\frac{S}{S^2+1}\right)V_2$$

$$\frac{V_1}{V_2} = s\left(\frac{S}{S^2+1}\right)$$

$$Hence, h_{12} = \frac{V_1}{V_2}\Big|_{I_1=0} = \frac{S^2}{S^2+1}$$

$$[h] = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

$$[h] = \begin{bmatrix} \frac{2S^2+1}{S(S^2+1)} & \frac{S^2}{S^2+1} \\ -\frac{S^2}{S^2+1} & \frac{S}{S^2+1} \end{bmatrix}$$

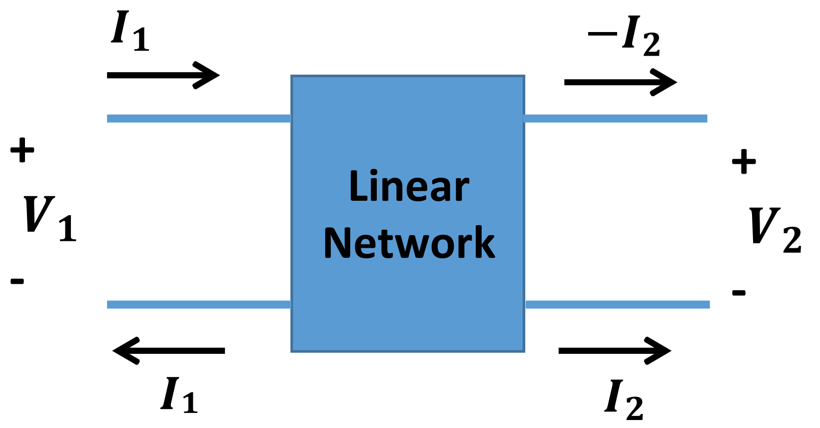
ABCD Parameters or Transmission Parameters

- Used in the analysis of power transmission in which input port is referred as the sending end while the output port is referred as the receiving end.
- There are obtained by expressing voltage $\boldsymbol{V_1}$ and current $\boldsymbol{I_1}$ at the input port in terms of voltage $\boldsymbol{V_2}$ and current $\boldsymbol{I_2}$ at the output port
- $\boldsymbol{V_2}$ *and* $\boldsymbol{I_2}$ are independent variables
- $\boldsymbol{V_1}$ *and* $\boldsymbol{I_1}$ are dependent variables

$$V_1 = f_1\,(V_2\,, -\,I_2\,)$$

$$I_1 = f_2\,(\,V_2\,, -\,I_2)$$

Above equations and be written as



$$V_1 = A\,V_2 + B\,(-I_2) \quad \ldots..eq\,(1)$$

$$I_1 = C\,V_2 + D\,(-\,I_2) \quad \ldots..eq\,(2)$$

In matrix form, equations can be written as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

ABCD Parameters or Transmission Parameters

- The individual ABCD Parameters can be obtained as follows

$$V_1 = A\,V_2 + B\,(-I_2)$$

$$I_1 = C\,V_2 + D\,(-I_2)$$

1. To obtain A

$$Let\ -I_2 = 0 \rightarrow Port\ 2\ is\ open\ circuited$$

From eq (1) $V_1 = A\,V_2$

$$\boxed{A = \left.\frac{V_1}{V_2}\right|_{-I_2=0}}$$

The parameter A is called open circuit reverse voltage gain. It is unitless

2. To obtain C

$$Let\ -I_2 = 0 \rightarrow Port\ 2\ is\ short\ circuited$$

From eq (2) $I_1 = C\,V_2$

$$\boxed{C = \left.\frac{I_1}{V_2}\right|_{-I_2=0} \quad \mho}$$

The parameter C is called open circuit reverse transfer admittance.

ABCD Parameters or Transmission Parameters

$V_1 = A V_2 + B(-I_2)$

$I_1 = C V_2 + D(-I_2)$

3. To obtain B

$Let\ V_2 = 0 \rightarrow Port\ 2\ is\ short\ circuited$

From eq (1) $V_1 = B(-)$

$$\boxed{B = \left.\frac{V_1}{-I_2}\right|_{V_2=0} \ \Omega}$$

The parameter B is called short circuit reverse transfer impedance.

4. To obtain D

$Let\ V_2 = 0 \rightarrow Port\ 2\ is\ short\ circuited$

From eq (2) $I_1 = D(-)$

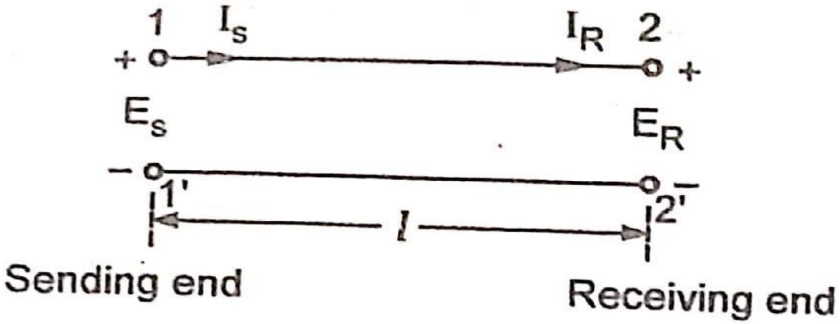
$$\boxed{D = \left.\frac{I_1}{-I_2}\right|_{V_2=0}}$$

The parameter D is called short circuit reverse current gain. It is unitless

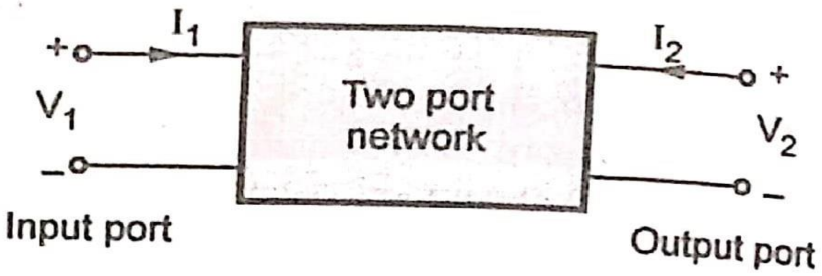
ABCD Parameters or Transmission Parameters

➢ **Reason for name transmission parameters and negative sign of I_2**

- ABCD parameters are used in the analysis of power transmission in which input port is referred as the **sending end** while the output port is referred as the **receiving end.**
- In transmission theory, the sending end variables are expressed in terms of receiving end variables.
- Analogous to the equations of the ABCD parameters. Hence the name **transmission parameters**.
- Also used for analysis of two or more networks connected in cascade. Hence it is also termed as **chain parameters.**



(a) Transmission line of length l



(b) Two port network

ABCD Parameters or Transmission Parameters

- From figure current direction of I_R is away from output port but in two port network I_2 it is towards output port. Hence it is assumed to be $-I_2$

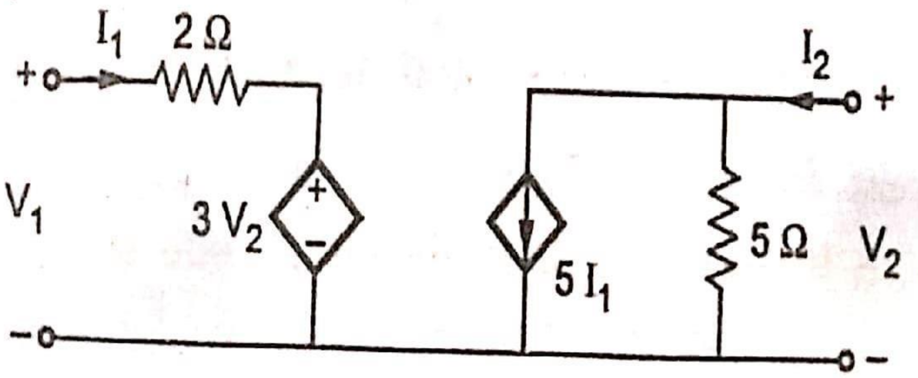
- **Conditions for Symmetry**

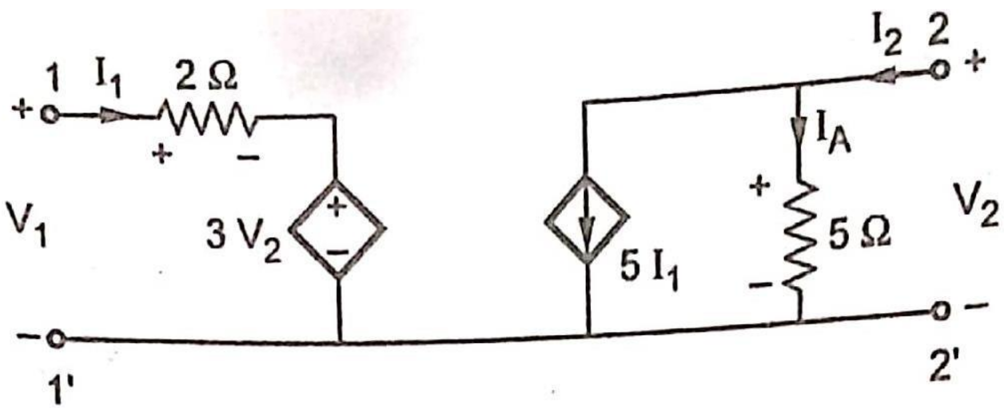
$$A = D$$

- **Conditions for Reciprocity**

$$AD - BC = 1$$

Problem 1: Determine the transmission parameters for the network shown in figure.





Transmission-parameters are given by

$V_1 = A V_2 + B(-I_2)$

$I_1 = C V_2 + D(-I_2)$

Apply KVL at the input side

$$-2I_1 - 3V_2 + V_1 = 0$$

$$V_1 = 2I_1 + 3V_2 \quad .... (1)$$

Apply KCL at the output side

$$I_2 = 5I_1 + I_A = 5I_1 + \frac{V_2}{5}$$

$$5I_1 = -\frac{V_2}{5} + I_2$$

$$I_1 = -\frac{V_2}{25} + \frac{1}{5} I_2 \quad .... (2)$$

Substitute eq (2) in eq(1)

$$V_1 = 2\left(-\frac{V_2}{25} + \frac{1}{5} I_2\right) + 3V_2$$

$$V_1 = \left(3 - \frac{2}{25}\right) V_2 + \frac{2}{5} I_2$$

$$V_1 = \left(\frac{73}{25}\right) V_2 + \frac{2}{5} I_2$$

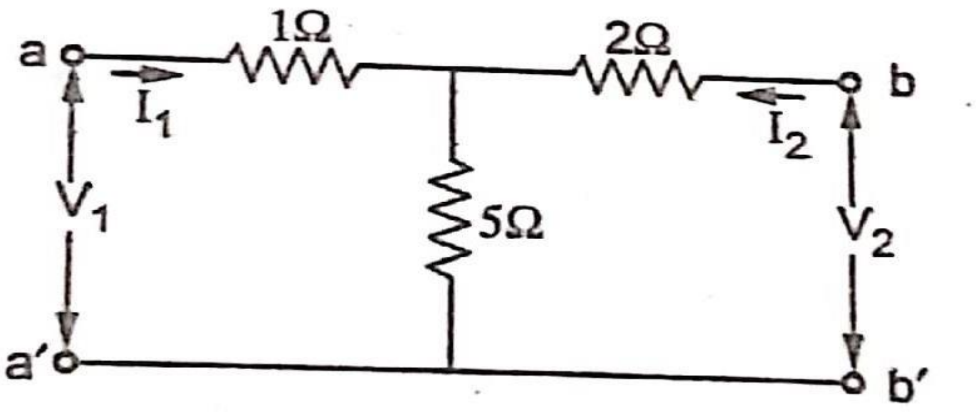
$$V_1 = \left(\frac{73}{25}\right) V_2 + \left(-\frac{2}{5}\right)(-I_2) \quad .... (3)$$

Rearranging eq (2)

$$I_1 = -\frac{V_2}{25} + \left(-\frac{1}{5}\right)(-I_2) \quad .... (4)$$

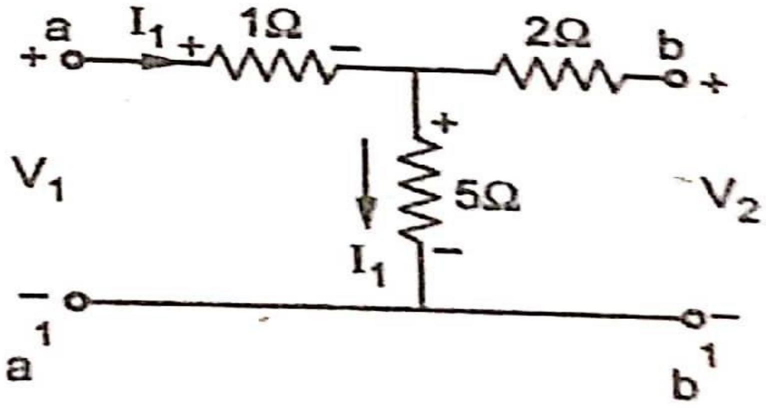
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{73}{25} & -\frac{2}{5} \\ -\frac{1}{25} & -\frac{1}{5} \end{bmatrix}$$

Problem 2: **Find the transmission or general parameters for the circuit shown in figure.**



Transmission-parameters are given by

$$V_1 = A\,V_2 + B\ (-I_2)$$
$$I_1 = C\,V_2 + D\,(-I_2)$$



To find transmission parameters

1. Let $-I_2 = 0 \rightarrow$ ***Port 2 is open circuited***

From Figure,

$$V_2 = 5I_1$$
$$I_1 = \frac{V_2}{5} \ldots (1)$$

Apply KVL at the input side

$$-1I_1 - 5I_1 + V_1 = 0$$
$$6I_1 = V_1 \quad \ldots.(2)$$

Substitute eq(1) in eq(2)

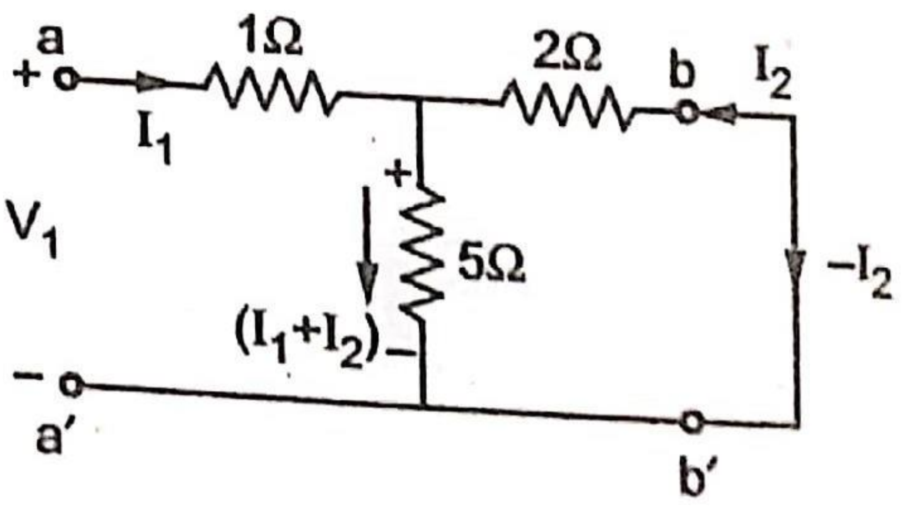
$$6\left(\frac{V_2}{5}\right) = V_1$$
$$V_1 = \frac{6}{5}V_2 \quad \ldots.(3)$$

$$From\ eq(1), C == \frac{I_1}{V_2}\Big|_{-I_2=0} = \frac{1}{5}\ ℧$$

$$From\ eq\ (3), A = \frac{V_1}{V_2}\Big|_{-I_2=0} = \frac{6}{5}$$

To find transmission parameters

$$2.\ Let\ V_2 = 0 \rightarrow Port\ 2\ is\ short\ circuited$$



From Figure, apply current divider rule

$$I_2 = -I_1\left(\frac{5}{5+2}\right)$$

$$-I_2 = I_1\left(\frac{5}{7}\right)$$

$$\frac{I_1}{I_2} = -\frac{7}{5} \quad \dots(4)$$

$$From\ eq\ (4),\ D = \left.\frac{I_1}{-I_2}\right|_{V_2=0} = \frac{7}{5}$$

Apply KVL at the input side

$$-1I_1 - 5(I_1+I_2) + V_1 = 0$$

$$-6I_1 - 5I_2 = -V_1 \quad \dots(5)$$

Substitute eq(4) I_1 for in eq(5)

$$6\left(-\frac{7}{5}\right)I_2 + 5I_2 = V_1$$

$$\left(-\frac{42}{5} + 5\right)I_2 = V_1$$

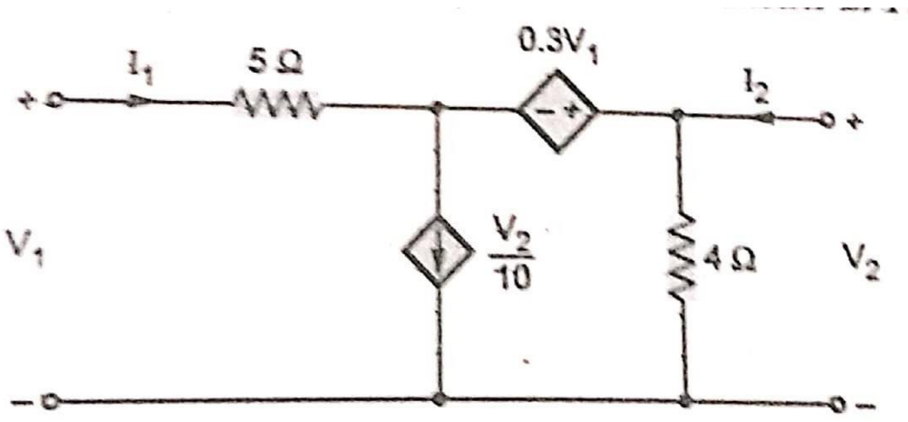
$$\left(-\frac{17}{5}\right)I_2 = V_1$$

$$\left(\frac{17}{5}\right)(-I_2) = V_1 \quad \dots(6)$$

$$From\ eq\ (6),\ B = \left.\frac{V_1}{-I_2}\right|_{V_2=0} = \frac{17}{5}\Omega$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{6}{5} & \frac{17}{5} \\ \frac{1}{5} & \frac{7}{5} \end{bmatrix}$$

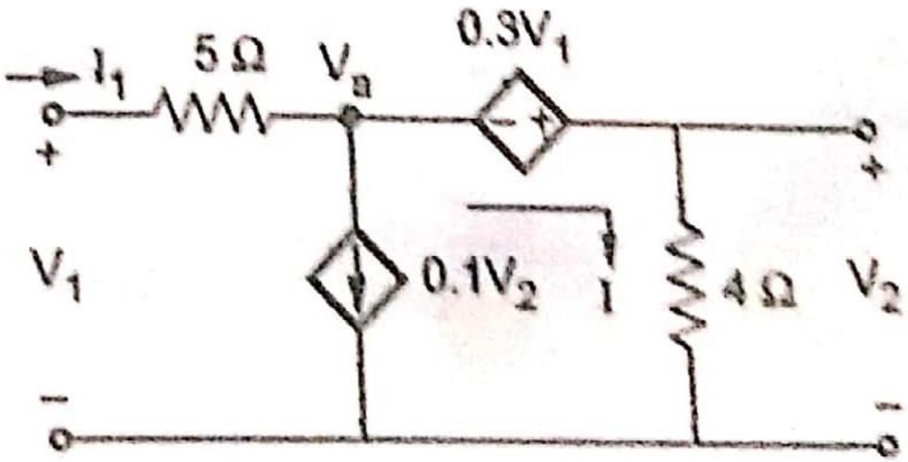
Problem 3: **Find the transmission parameters for the network shown in figure.**



Transmission-parameters are given by

$V_1 = A\,V_2 + B\,(-I_2)$

$I_1 = C\,V_2 + D\,(-I_2)$



To find transmission parameters

$1.\ Let\ -I_2 = 0 \ \rightarrow Port\ 2\ is\ open\ circuited$

Apply KCL at node a

$I_1 = 0.1V_2 + I$

$I = I_1 - 0.1V_2$

From figure

$V_2 = 4I = 4(I_1 - 0.1V_2)$

$1.4\,V_2 = 4\,I_1 \ \ldots(1)$

$From\ eq\ (1),\ C = \left.\frac{I_1}{V_2}\right|_{-I_2=0} = \frac{7}{20}\ ℧$

Apply KVL at the input side

$V_1 - 5I_1 - V_a = 0$

$V_a = V_1 - 5I_1 \quad \ldots.(2)$

From figure

$V_a + 0.3V_1 = V_2 \quad \ldots(4)$

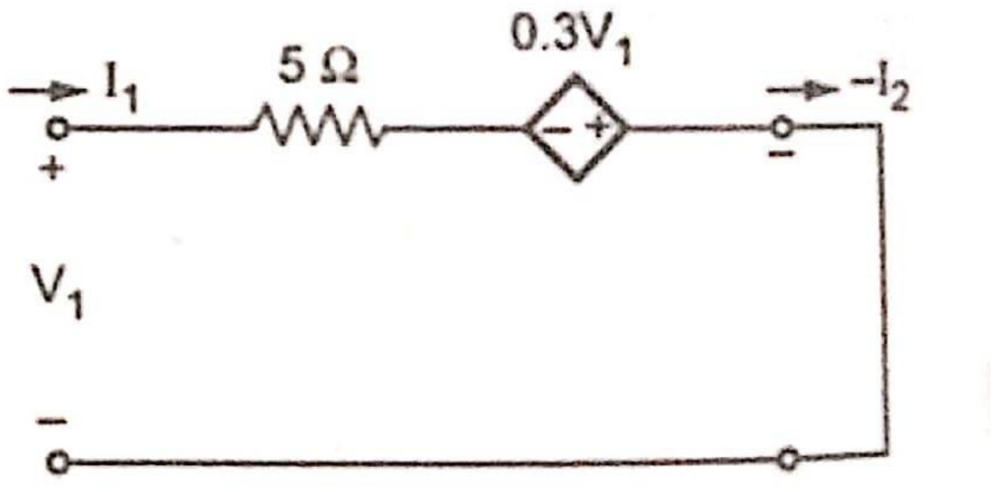
Substitute eq(1) for I_1 in eq (2)

$V_a = V_1 - 5\left(\frac{7}{20}V_2\right) = V_1 - \frac{7}{4}V_2 \quad \ldots(3)$

Substitute for Va from eq(3) in eq(4)

$$V_1 - \frac{7}{4}V_2 + 0.3V_1 = V_2$$

$$1.3V_1 = \frac{11}{4}V_2 \quad \dots (5)$$



$$From\ eq\ (5), A = \left.\frac{V_1}{V_2}\right|_{-I_2=0} = \frac{55}{26}$$

To find transmission parameters

$$2.\ Let\ V_2 = 0 \rightarrow Port\ 2\ is\ short\ circuited$$

$$Hence\ 4\Omega\ get\ short\ circuited\ and\ 0.1V_2\ get\ open\ circuited$$

From figure

$$-I_2 = I_1 \quad \dots (6)$$

$$From\ eq\ (4), D = \left.\frac{I_1}{-I_2}\right|_{V_2=0} = 1$$

Apply KVL to the loop

$$V_1 - 5I_1 + 0.3V_1 = 0$$

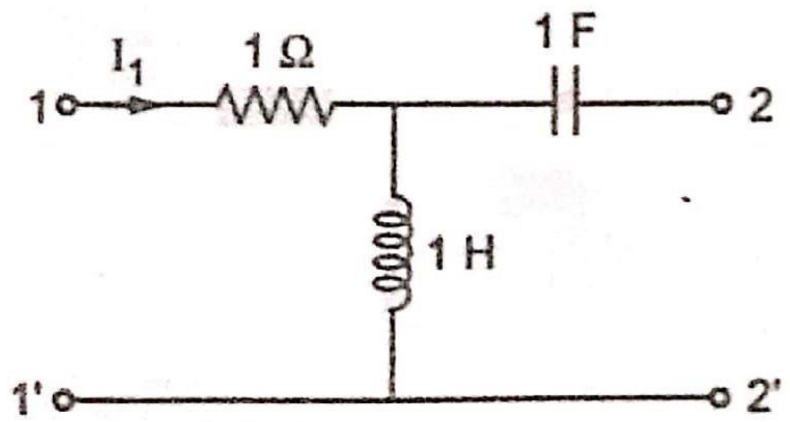
$$I_1 = \frac{1.3V_1}{5} = \frac{13V_1}{50} = -I_2$$

$$-I_2 = \frac{13}{50}V_1 \quad \dots (7)$$

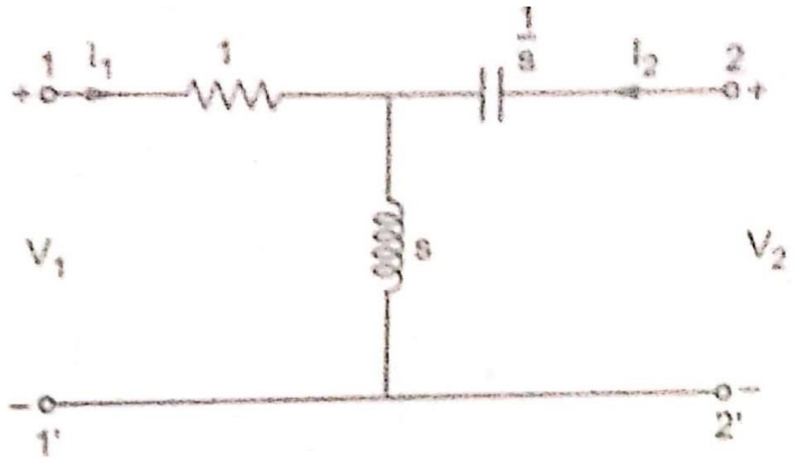
$$From\ eq(7), B = \left.\frac{V_1}{-I_2}\right|_{V_2=0} = \frac{50}{13}\Omega$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{55}{26} & \frac{50}{13} \\ \frac{7}{20} & 1 \end{bmatrix}$$

Problem 4: Find the transmission parameters for the network shown in figure.



Transformed network



Transmission-parameters are given by

$$V_1 = A\,V_2 + B\,(-I_2)$$
$$I_1 = C\,V_2 + D\,(-I_2)$$

To find transmission parameters

1. $Let\ -I_2 = 0 \ \rightarrow Port\ 2\ is\ open\ circuited$

From figure

$$V_2 = s\,I_1 \quad \ldots\ (1)$$

$$From\ eq\ (1), C == \frac{I_1}{V_2}\Big|_{-I_2=0} = \frac{1}{s}\ \mho$$

Apply KVL at the input side

$$V_1 - 1I_1 - sI_1 = 0$$
$$V_1 = (1 + s)I_1 \quad \ldots (2)$$

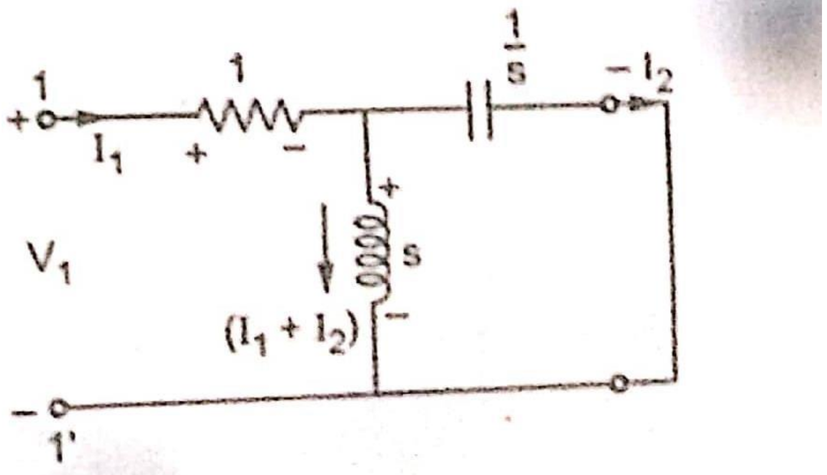
Substitute for I_1 from eq (1) in eq(2)

$$V_1 = (1+s)\left(\frac{V_2}{s}\right)$$
$$\frac{V_1}{V_2} = \frac{(1+s)}{s} \quad \ldots (3)$$

$$From\ eq\ (3), A = \frac{V_1}{V_2}\Big|_{-I_2=0} = \frac{(1+s)}{s}$$

To find transmission parameters

2. $Let\ V_2 = 0 \rightarrow Port\ 2\ is\ short\ circuited$



From Figure, apply current divider rule

$$-I_2 = I_1\left(\frac{S}{S+\frac{1}{S}}\right) = \left(\frac{S^2}{S^2+1}\right) I_1 \quad \dots (4)$$

$$From\ eq\ (4), D = \left.\frac{I_1}{-I_2}\right|_{V_2=0} = \frac{S^2+1}{S^2}$$

Apply KVL at the input side

$$V_1 - 1I_1 - S\,(I_1 + I_2) = 0$$

$$V_1 = (1 + S)I_1 + SI_2 \dots\dots (5)$$

Substitute for I_1 from eq (4) in eq(5)

$$V_1 = (1 + S)\left(-\frac{S^2+1}{S^2}\right) + SI_2$$

$$V_1 = (1 + S)\left(-\frac{S^2+1}{S^2}\right) - S(-I_2)$$

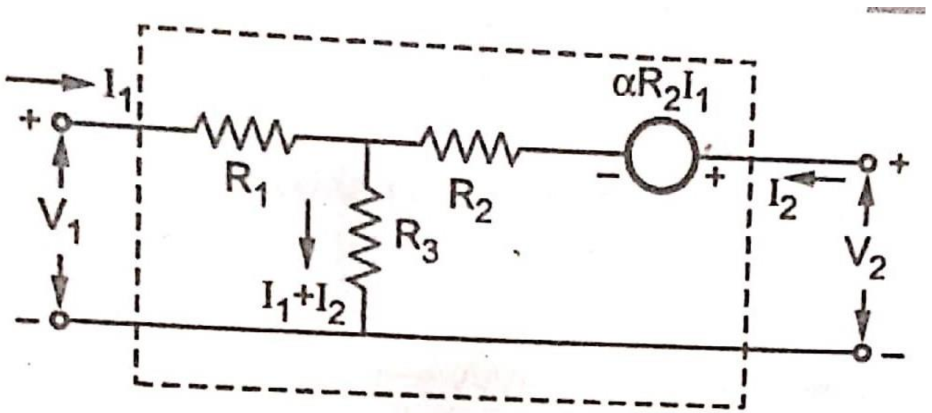
$$V_1 = \left(\frac{1 + S + S^2 + S^3 - S^3}{S^2}\right)(-I_2)$$

$$V_1 = \left(\frac{1 + S + S^2}{S^2}\right)(-I_2)$$

$$From\ eq\ (6),\ B = \left.\frac{V_1}{-I_2}\right|_{V_2=0} = \frac{1+S+S^2}{S^2}\Omega$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{(1+s)}{s} & \frac{1+S+S^2}{S^2} \\ \frac{1}{s} & \frac{S^2+1}{S^2} \end{bmatrix}$$

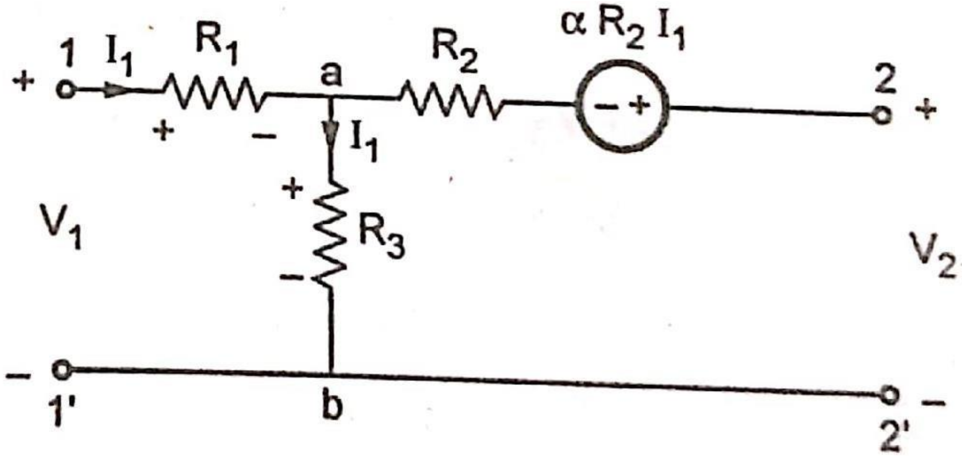
Problem 5: For the network shown in figure, determine the ABCD parameters.



Transmission-parameters are given by

$V_1 = A\,V_2 + B\ (-I_2)$

$I_1 = C\,V_2 + D\,(-I_2)$



To find transmission parameters

1. Let $-I_2 = 0 \rightarrow$ Port 2 is open circuited

Applying KVL to closed path 1−a−b−1′−a, we get,

$$-I_1 R_1 - I_1 R_3 + V_1 = 0$$

$$\therefore \quad V_1 = (R_1 + R_3) I_1 \quad \text{... (1)}$$

At output side,

$$V_2 = \alpha R_2 I_1 + I_1 R_3 = (R_3 + \alpha R_2) I_1$$

$$I_1 = \frac{V_2}{(R_3 + \alpha R_2)} \quad \text{... (2)}$$

Hence from equation (2),

$$C = \left.\frac{I_1}{V_2}\right|_{-I_2=0} = \frac{1}{(R_3 + \alpha R_2)}\ \mho$$

$$V_1 = (R_1+R_3)\left[\frac{V_2}{(R_3+\alpha R_2)}\right] = \left[\frac{R_1+R_3}{R_3+\alpha R_2}\right]V_2 \quad \text{... (3)}$$

To find transmission parameters

2. Let $V_2 = 0 \rightarrow Port\ 2\ is\ short\ circuited$

Applying KVL to closed path, 1-a-b-1′-1, we get,

$$-I_1 R_1 - R_3 (I_1 + I_2) + V_1 = 0$$

$$\therefore \quad V_1 = (R_1+R_3)I_1 + R_2 I_2 \quad \text{... (4)}$$

Applying KVL to closed path a-2-2′-b-a, we get,

$$+I_2 R_2 + \alpha R_2 I_1 + R_3 (I_1 + I_2) = 0$$

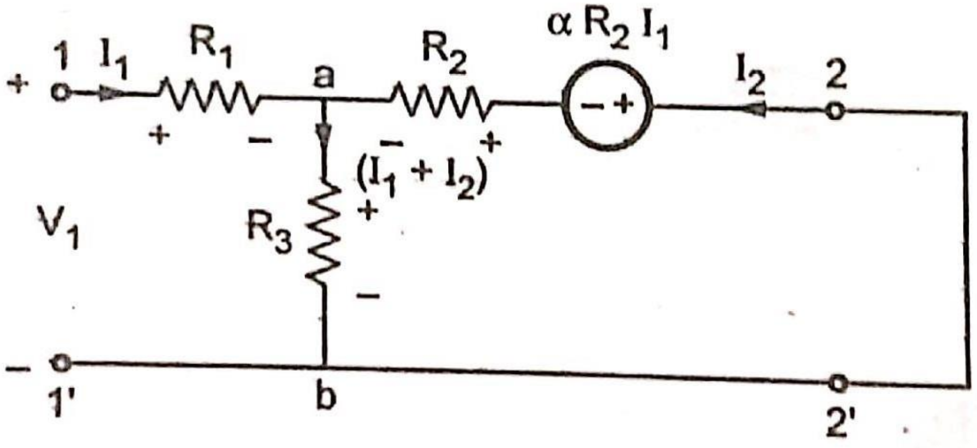
$$\therefore I_2 R_2 + a R_2 I_1 + I_1 R_3 + I_2 R_3 = 0$$

$$\therefore [R_3 + \alpha R_2] I_1 = -(R_2+R_3) I_2$$

$$\text{i.e.} (R_3 + \alpha R_2) I_1 = (R_2+R_3)(-I_2) \quad \text{... (5)}$$

from equation (3),

$$A = \left.\frac{V_1}{V_2}\right|_{-I_2=0} = \frac{R_1+R_3}{(R_3+\alpha R_2)}$$



From equation (5),

$$I_1 = \frac{R_2+R_3}{R_3+\alpha R_2}(-I_2) \quad \text{... (6)}$$

from equation (6) we get,

$$D = \left.\frac{I_1}{-I_2}\right|_{V_2=0} = \frac{R_2+R_3}{R_3+\alpha R_2}$$

Substitute for I_1 from eq (6) in eq(4)

$$V_1 = (R_1 + R_3)\left[\frac{R_2 + R_3}{R_3 + \alpha R_2}\right](-I_2) + R_2 I_2$$

$$V_1 = \frac{(R_1 + R_3)(R_2 + R_3)}{R_3 + \alpha R_2}(-I_2) - R_2(-I_2)$$ ... Adjusting $-I_2$ in 2nd term

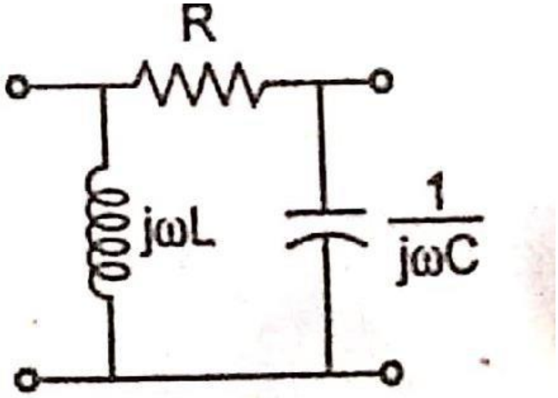
$$V_1 = \left[\frac{R_1 R_2 + R_1 R_3 + R_2 R_3 + R_3^2 - R_2 R_3 - \alpha R_2^2}{R_3 + \alpha R_2}\right](-I_2)$$

$$V_1 = \left[\frac{R_1 R_2 + R_1 R_3 + R_3^2 - \alpha R_2^2}{R_3 + \alpha R_2}\right](-I_2) \qquad \text{... (7)}$$

from equation (7), we get,

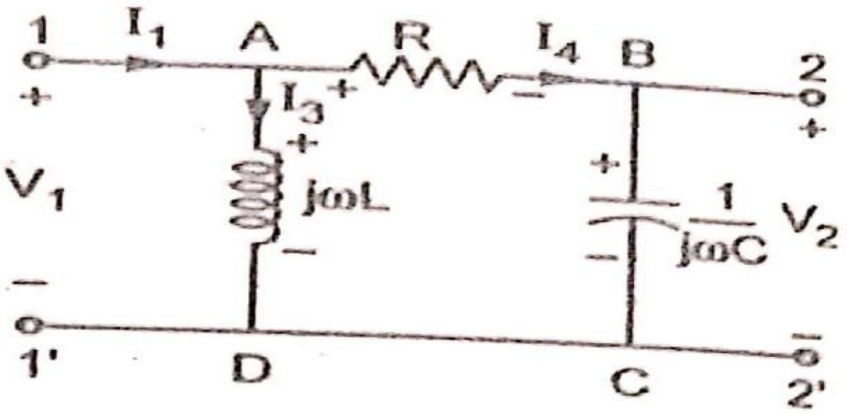
$$A = \left.\frac{I_1}{-I_2}\right|_{V_2=0} = \frac{R_1 R_2 + R_1 R_3 + R_3^2 - \alpha R_2^2}{R_3 + \alpha R_2}\ \Omega$$

Problem 6: Determine the transmission parameters of the network shown.



Transmission-parameters are given by

$$V_1 = A\,V_2 + B\,(-I_2)$$
$$I_1 = C\,V_2 + D\,(-I_2)$$



To find transmission parameters

1. Let $-I_2 = 0 \rightarrow$ Port 2 is open circuited

Applying KVL to loop 1-A-D-1′-1, we get,

$$(-j\omega L)\,I_3 + V_1 = 0$$

$$V_1 = (j\omega L)\,I_3$$

Applying KVL to loop A-B-C-D-A, we get,

$$-R\,I_4 - \left(\frac{1}{j\omega C}\right)I_4 + j\omega L\,I_3 = 0$$

$$(j\omega L)\,I_3 = \left[R + \frac{1}{j\omega C}\right]I_4 = \left[\frac{j\omega RC + 1}{j\omega C}\right]I_4$$

Applying current divider rule to get I_4

$$I_4 = I_1\left[\frac{j\omega L}{R + j\omega L + \frac{1}{j\omega C}}\right] = \left[\frac{-\omega^2 LC}{1 - \omega^2 LC + j\omega RC}\right]I_1$$

Also $$V_2 = \left(\frac{1}{j\omega C}\right) I_4 = \left(\frac{1}{j\omega C}\right)\left[\frac{-\omega^2 LC}{1-\omega^2 LC + j\omega RC}\right] I_1 = \left[\frac{j\omega L}{1-\omega^2 LC + j\omega RC}\right] I_1 \quad \ldots (1)$$

From eq (1)

$$\therefore \quad C = \left.\frac{I_1}{V_2}\right|_{-I_2=0} = \frac{1-\omega^2 LC + j\omega RC}{j\omega L}$$

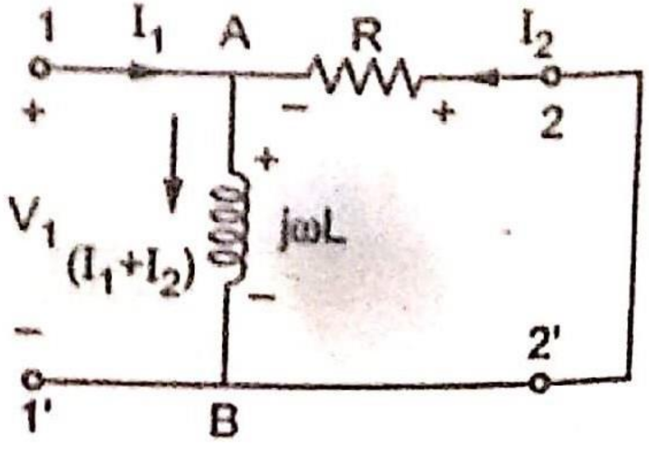
$$V_1 = j\omega L\,(I_3) = j\omega L\left[\frac{j\omega RC+1}{(j\omega C)\,(J\omega L)}\right] I_4 = \left[\frac{j\omega RC+1}{j\omega C}\right]\left[\frac{-\omega^2 LC}{1-\omega^2 LC + j\omega RC}\right] I_1 \quad \ldots (2)$$

Dividing equation (2) by (1),

$$A = \left.\frac{V_1}{V_2}\right|_{-I_2=0} = \left[\frac{j\omega RC+1}{j\omega C}\right]\left[\frac{-\omega^2 LC}{1-\omega^2 LC + j\omega RC}\right]\left[\frac{1-\omega^2 LC + j\omega RC}{j\omega L}\right] = \frac{1+j\omega RC}{1-\omega^2 LC + j\omega RC}$$

To find transmission parameters

2. Let $V_2 = 0 \rightarrow$ Port 2 is short circuited



Applying KVL to loop 1-A-B-1′-1, we get,

$$V_1 = j\omega L\,(I_1 + I_2) = j\omega L\, I_1 + j\omega L\, I_2 \quad \text{... (3)}$$

Applying KVL to loop A-2-2′-B-A, we get,

$$R I_2 + j\omega L I_1 + j\omega L\, I_2 = 0$$

$$j\omega L\, I_1 + (R + j\omega L)\, I_2 = 0$$

$$j\omega L\, I_1 = -(R + j\omega L)\, I_2 = (R + j\omega L)\,(-I_2) \quad \text{... (4)}$$

$$D = \left.\frac{I_1}{-I_2}\right|_{V_2=0} = \frac{R + j\omega L}{j\omega L}$$

From equation (4), I_1 is given by,

$$I_1 = \left[\frac{R + j\omega L}{j\omega L}\right](-I_2)$$

Putting value of I_1 in equation (3),

$$V_1 = j\omega L\left[\frac{R + j\omega L}{j\omega L}\right](-I_2) + j\omega L\, I_2$$

$$V_1 = (R + j\omega L)\,(-I_2) - j\omega L\,(-I_2)$$

$$B = \left.\frac{V_1}{-I_2}\right|_{V_2=0} = R$$

Summary of Two Port Network Parameters

Parameters	Variables		Equations	Conditions of	
	Dependent	Independent		Symmetry	Reciprocity
z parameter	$V_1 and\ V_2$	$I_1 and\ I_2$	$V_1 = z_{11} I_1 + z_{12} I_2$ $V_2 = z_{21} I_1 + z_{22} I_2$	$z_{11} = z_{22}$	$z_{12} = z_{21}$
y parameter	$I_1 and\ I_2$	$V_1 and\ V_2$	$I_1 = y_{11} V_1 + y_{12} V_2$ $I_2 = y_{21} V_1 + y_{22} V_2$	$y_{11} = y_{22}$	$y_{12} = y_{21}$
h parameter	$V_1 and\ I_2$	$I_1\ and\ V_2$	$V_1 = h_{11} I_1 + h_{12} V_2$ $I_2 = h_{21} I_1 + h_{22} V_2$	$h_{11} h_{22} - h_{12} h_{21} = 1$	$h_{12} = -h_{21}$
ABCD parameter	$V_1\ and\ I_1$	$V_2\ and\ I_2$	$V_1 = A V_2 + B (-I_2)$ $I_1 = C V_2 + D (-I_2)$	$A = D$	$AD - BC = 1$

Interrelationships Between Parameters

z-parameters in terms of other parameters

The equations for z-parameters are as follows

$$V_1 = z_{11} I_1 + z_{12} I_2 \quad \dots (A)$$

$$V_2 = z_{21} I_1 + z_{22} I_2 \quad \dots (B)$$

[A] In terms of y-parameters

The equations for y-parameters are as follows

$$I_1 = y_{11} V_1 + y_{12} V_2 \quad .. (1)$$

$$I_2 = y_{21} V_1 + y_{22} V_2 \quad .. (2)$$

Writing Equations in Matrix Form

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Solve equations using Cramer's rule for V_1 and V_2

$$\Delta = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} I_1 & y_{12} \\ I_2 & y_{22} \end{vmatrix}$$

$$V_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} I_1 & y_{12} \\ I_2 & y_{22} \end{vmatrix}}{\begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}}$$

$$V_1 = \frac{I_1 y_{22} - I_2 y_{12}}{y_{11} y_{22} - y_{12} y_{21}}$$

Let, $\Delta y = y_{11} y_{22} - y_{12} y_{21}$

$$V_1 = \frac{y_{22}}{\Delta y} I_1 + \frac{-y_{12}}{\Delta y} I_2 \dots (3)$$

$$V_1 = \frac{y_{22}}{\Delta y} I_1 + \frac{-y_{12}}{\Delta y} I_2 \ldots (3)$$

$$V_1 = z_{11} I_1 + z_{12} I_2$$
$$V_2 = z_{21} I_1 + z_{22} I_2$$

Similarly

$$V_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} y_{11} & I_1 \\ y_{21} & I_2 \end{vmatrix}}{\begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}}$$

$$V_2 = \frac{I_2 y_{11} - I_1 y_{21}}{y_{11} y_{22} - y_{12}\, y_{21}}$$

$$V_2 = \frac{-y_{21}}{\Delta y} I_1 + \frac{y_{11}}{\Delta y} I_2 \; .. (4)$$

Matrix form of z parameters

$$z = \begin{bmatrix} \frac{y_{22}}{\Delta y} & \frac{-y_{12}}{\Delta y} \\ \frac{-y_{21}}{\Delta y} & \frac{y_{11}}{\Delta y} \end{bmatrix}$$

Comparing equations (3) and (4) with equations (A) and (B) we get

$$z_{11} = \frac{y_{22}}{\Delta y} \qquad z_{12} = \frac{-y_{12}}{\Delta y}$$

$$z_{21} = \frac{-y_{21}}{\Delta y} \qquad z_{22} = \frac{y_{11}}{\Delta y}$$

Interrelationships Between Parameters

Z-parameters in terms of other parameters

The equations for z-parameters are as follows

$$V_1 = z_{11} I_1 + z_{12} I_2 \quad \ldots (A)$$

$$V_2 = z_{21} I_1 + z_{22} I_2 \quad \ldots (B)$$

[B] In terms of h-parameters

The equations for h-parameters are as follows

$$V_1 = h_{11} I_1 + h_{12} V_2 \quad \ldots\ldots (1)$$

$$I_2 = h_{21} I_1 + h_{22} V_2 \quad \ldots\ldots (2)$$

From eq(2) we can write

$$h_{22} V_2 = -h_{21} I_1 + I_2$$

$$V_2 = \frac{-h_{21}}{h_{22}} I_1 + \frac{1}{h_{22}} I_2 \quad \ldots (3)$$

Substitute V_2 in eq (1)

$$V_1 = h_{11} I_1 + h_{12} \left(\frac{-h_{21}}{h_{22}} I_1 + \frac{1}{h_{22}} I_2 \right)$$

$$V_1 = \left(h_{11} - \frac{h_{12}\, h_{21}}{h_{22}} \right) I_1 + \frac{h_{12}}{h_{22}} I_2$$

$$V_1 = \frac{h_{11} h_{22} - h_{12} h_{21}}{h_{22}} I_1 + \frac{h_{12}}{h_{22}} I_2$$

$$V_1 = \frac{\Delta h}{h_{22}} I_1 + \frac{h_{12}}{h_{22}} I_2 \quad \ldots (4)$$

Comparing equations (3) and (4) with equations (A) and (B) we get

$$z_{11} = \frac{\Delta h}{h_{22}} \qquad z_{12} = \frac{h_{12}}{h_{22}}$$

$$z_{21} = \frac{-h_2}{h_{22}} \qquad z_{22} = \frac{1}{h_{22}}$$

Matrix form of z parameters

$$z = \begin{bmatrix} \frac{\Delta h}{h_{22}} & \frac{h_{12}}{h_{22}} \\ \frac{-h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{bmatrix}$$

Interrelationships Between Parameters

Z-parameters in terms of other parameters

The equations for z-parameters are as follows

$$V_1 = z_{11} I_1 + z_{12} I_2 \quad \dots (A)$$

$$V_2 = z_{21} I_1 + z_{22} I_2 \quad \dots (B)$$

[C] In terms of Transmission(ABCD) parameters

The equations for ABCD-parameters are as follows

$$V_1 = A V_2 + B\,(-I_2) \quad \dots\dots (1)$$

$$I_1 = C V_2 + D\,(-I_2) \quad \dots\dots (2)$$

From eq(2) we can write

$$C V_2 = I_1 + D\,(I_2)$$

$$V_2 = \frac{1}{C} I_1 + \frac{D}{C} I_2 \quad ..(3)$$

Substitute V_2 in eq (1)

$$V_1 = A\left(\frac{1}{C} I_1 + \frac{D}{C} I_2\right) + B\,(-I_2)$$

$$V_1 = \frac{A}{C} I_1 + \left(\frac{AD}{C} - B\right) I_2$$

$$V_1 = \frac{A}{C} I_1 + \left(\frac{AD - BC}{C}\right) I_2$$

$$V_1 = \frac{A}{C} I_1 + \left(\frac{\Delta T}{C}\right) I_2 \quad ..(4)$$

Comparing equations (3) and (4) with equations (A) and (B) we get

$$z_{11} = \frac{A}{C} \qquad z_{12} = \frac{\Delta T}{C}$$

$$z_{21} = \frac{1}{C} \qquad z_{22} = \frac{D}{C}$$

Matrix form of z parameters

$$z = \begin{bmatrix} \frac{A}{C} & \frac{\Delta T}{C} \\ \frac{1}{C} & \frac{D}{C} \end{bmatrix}$$

Interrelationships Between Parameters

y-parameters in terms of other parameters

The equations for y-parameters are as follows

$$I_1 = y_{11} V_1 + y_{12} V_2 \quad .. (A)$$

$$I_2 = y_{21} V_1 + y_{22} V_2 \quad .. (B)$$

[A] In terms of z-parameters

The equations for z-parameters are as follows

$$V_1 = z_{11} I_1 + z_{12} I_2 \quad ... (1)$$

$$V_2 = z_{21} I_1 + z_{22} I_2 \quad ... (2)$$

Writing Equations in Matrix Form

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Solve equations using Cramer's rule for I_1 and I_2

$$\Delta = \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} V_1 & z_{12} \\ V_2 & z_{22} \end{vmatrix}$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} V_1 & z_{12} \\ V_2 & z_{22} \end{vmatrix}}{\begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}}$$

$$I_1 = \frac{V_1 z_{22} - V_2 z_{12}}{z_{11} z_{22} - z_{12} z_{21}}$$

$$\Delta z = z_{11} z_{22} - z_{12} z_{21}$$

$$I_1 = \frac{z_{22}}{\Delta z} V_1 + \frac{-z_{12}}{\Delta z} V_2 \ ... (3)$$

$$I_1 = \frac{z_{22}}{\Delta z} V_1 + \frac{-z_{12}}{\Delta z} V_2 \;\ldots (3)$$

$$I_1 = y_{11} V_1 + y_{12} V_2$$

$$I_2 = y_{21} V_1 + y_{22} V_2$$

Similarly

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} z_{11} & V_1 \\ z_{21} & V_2 \end{vmatrix}}{\begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}}$$

$$I_2 = \frac{V_2 z_{11} - V_1 z_{21}}{z_{11} z_{22} - z_{12}\, z_{21}}$$

$$I_2 = \frac{-z_{21}}{\Delta z} V_1 + \frac{z_{11}}{\Delta z} V_2 \;\ldots (4)$$

Matrix form of y parameters

$$y = \begin{bmatrix} \frac{z_{22}}{\Delta z} & \frac{-z_{12}}{\Delta z} \\ \frac{-z_{21}}{\Delta z} & \frac{z_{11}}{\Delta z} \end{bmatrix}$$

Comparing equations (3) and (4) with equations (A) and (B) we get

$$y_{11} = \frac{z_{22}}{\Delta z} \qquad y_{12} = \frac{-z_{12}}{\Delta z}$$

$$y_{21} = \frac{-z_{21}}{\Delta z} \qquad y_{22} = \frac{z_{11}}{\Delta z}$$

Interrelationships Between Parameters

y-parameters in terms of other parameters

The equations for y-parameters are as follows

$$I_1 = y_{11} V_1 + y_{12} V_2 \quad .. \ (A)$$

$$I_2 = y_{21} V_1 + y_{22} V_2 \quad .. (B)$$

[B] In terms of h-parameters

The equations for h-parameters are as follows

$$V_1 = h_{11} I_1 + h_{12} V_2 \quad ..... \ (1)$$

$$I_2 = h_{21} I_1 + h_{22} V_2 \quad ..... \ (2)$$

From eq(1) we can write

$$h_{11} I_1 = V_1 - h_{12} V_2$$

$$I_1 = \frac{1}{h_{11}} V_1 + \frac{-h_{12}}{h_{11}} V_2 \ .. (3)$$

Substitute I_1 in eq (2)

$$I_2 = h_{21} \left(\frac{1}{h_{11}} V_1 + \frac{-h_{12}}{h_{11}} V_2 \right) + h_{22} V_2$$

$$I_2 = \frac{h_{21}}{h_{11}} V_1 + \left(h_{22} - \frac{h_{21} h_{12}}{h_{11}}\right) V_2$$

$$I_2 = \frac{h_{21}}{h_{11}} V_1 + \left(\frac{h_{22} h_{11} - h_{21} h_{12}}{h_{11}} \right) V_2$$

$$I_2 = \frac{h_{21}}{h_{11}} V_1 + \left(\frac{\Delta h}{h_{11}} \right) V_2$$

Comparing equations (3) and (4) with equations (A) and (B) we get

$$y_{11} = \frac{1}{h_{11}} \qquad y_{12} = \frac{-h_{12}}{h_{11}}$$

$$y_{21} = \frac{h_{21}}{h_{11}} \qquad y_{22} = \frac{\Delta h}{h_{11}}$$

Matrix form of y parameters

$$y = \begin{bmatrix} \frac{1}{h_{11}} & \frac{-h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\Delta h}{h_{11}} \end{bmatrix}$$

Interrelationships Between Parameters

y-parameters in terms of other parameters

The equations for y-parameters are as follows

$$I_1 = y_{11} V_1 + y_{12} V_2 \quad .. (A)$$

$$I_2 = y_{21} V_1 + y_{22} V_2 \quad ..(B)$$

[C] In terms of Transmission(ABCD) parameters

The equations for y-parameters are as follows

$$V_1 = A V_2 + B\ (-I_2) \quad ..... (1)$$

$$I_1 = C V_2 + D\ (-I_2) \quad ..... (2)$$

From eq(1) we can write

$$B\ (-I_2) = V_1 - A V_2$$

$$I_2 = \frac{-1}{B} V_1 + \frac{A}{B} V_2 ... (3)$$

Substitute I_2 in eq (2)

$$I_1 = C V_2 + D \left(\frac{1}{B} V_1 - \frac{A}{B} V_2\right)$$

$$I_1 = \frac{D}{B} V_1 + \left(C - \frac{AD}{B}\right) V_2$$

$$I_1 = \frac{D}{B} V_1 + \left(\frac{BC - AD}{B}\right) V_2$$

$$I_1 = \frac{D}{B} V_1 + \left(\frac{\Delta T}{B}\right) V_2 ... \quad \textbf{(4)}$$

Comparing equations (3) and (4) with equations (A) and (B) we get

$$y_{11} = \frac{D}{B} \qquad y_{12} = \frac{-\Delta T}{B}$$

$$y_{21} = \frac{-1}{B} \qquad y_{22} = \frac{A}{B}$$

Matrix form of y parameters

$$y = \begin{bmatrix} \frac{D}{B} & \frac{-\Delta T}{B} \\ \frac{-1}{B} & \frac{A}{B} \end{bmatrix}$$

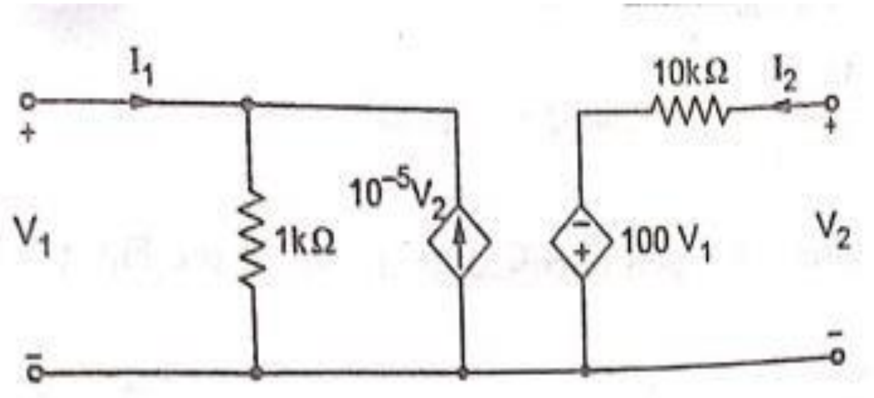
Interrelationships Between Parameters

1. h-parameters in terms of other parameters **(Assignment)**
2. Transmission parameters(ABCD) in terms of other parameters **(Assignment)**

Interrelationships between all the parameters

	[z]	[y]	[h]	[T]
[z]	$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$	$\begin{bmatrix} \frac{y_{22}}{\Delta y} & \frac{-y_{12}}{\Delta y} \\ \frac{-y_{21}}{\Delta y} & \frac{y_{11}}{\Delta y} \end{bmatrix}$	$\begin{bmatrix} \frac{\Delta h}{h_{22}} & \frac{h_{12}}{h_{22}} \\ \frac{-h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{bmatrix}$	$\begin{matrix} \frac{A}{C} & \frac{\Delta T}{C} \\ \frac{1}{C} & \frac{D}{C} \end{matrix}$
[y]	$\begin{bmatrix} \frac{z_{22}}{\Delta z} & \frac{-z_{12}}{\Delta z} \\ \frac{-z_{21}}{\Delta z} & \frac{z_{11}}{\Delta z} \end{bmatrix}$	$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$	$\begin{matrix} \frac{1}{h_{11}} & \frac{-h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\Delta h}{h_{11}} \end{matrix}$	$\begin{matrix} \frac{D}{B} & \frac{-\Delta T}{B} \\ \frac{-1}{B} & \frac{A}{B} \end{matrix}$
[h]	$\begin{bmatrix} \frac{\Delta z}{z_{22}} & \frac{z_{21}}{z_{22}} \\ \frac{-z_{21}}{z_{22}} & \frac{1}{z_{22}} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{y_{11}} & \frac{-y_{12}}{y_{11}} \\ \frac{y_{21}}{y_{11}} & \frac{\Delta y}{y_{11}} \end{bmatrix}$	$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$	$\begin{matrix} \frac{B}{D} & \frac{\Delta T}{D} \\ \frac{-1}{D} & \frac{C}{D} \end{matrix}$
[T]	$\begin{bmatrix} \frac{z_{11}}{z_{21}} & \frac{\Delta z}{z_{21}} \\ \frac{1}{z_{21}} & \frac{z_{22}}{z_{21}} \end{bmatrix}$	$\begin{bmatrix} -\frac{y_{22}}{y_{21}} & \frac{-1}{y_{21}} \\ -\frac{\Delta y}{y_{21}} & -\frac{y_{11}}{y_{21}} \end{bmatrix}$	$\begin{bmatrix} -\frac{\Delta h}{h_{21}} & \frac{-h_{11}}{h_{21}} \\ \frac{-h_{22}}{h_{21}} & \frac{-1}{h_{21}} \end{bmatrix}$	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

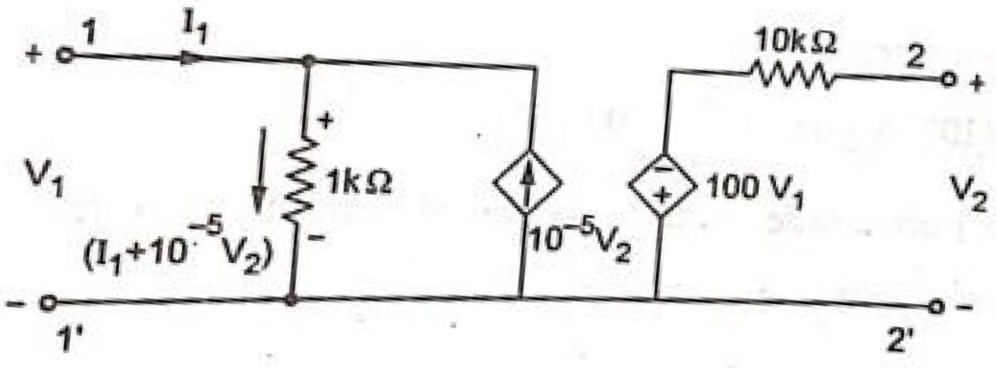
Problem 1: Find [z] for the two port network shown in figure. From [z] parameters calculate [y] parameters.



Z-parameters are given by

$$V_1 = z_{11} I_1 + z_{12} I_2$$

$$V_2 = z_{21} I_1 + z_{22} I_2$$



To find z-parameters

$$1. \, Let\ I_2 = 0 \quad \rightarrow port\ 2\ open\ circuited$$

From fig $V_1 = 1 * 10^3 \left(I_1 + 10^{-5} V_2\right) .. (1)$

At Port 2

$$V_2 = -100V_1 .. (2) \text{ (no drop across 10k}\Omega)$$

Substitute for V_2 **from eq (2) in eq(1)**

$$V_1 = 1 * 10^3 \left(I_1 + 10^{-5} (-100V_1)\right)$$

$$V_1 = 1000 \ I_1 - V_1$$

$$2V_1 = 1000 \ I_1$$

$$V_1 = 500 \ I_1 \ .. (3)$$

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = 500\ \Omega$$

Substitute $for\ V_1$ **from** $eq(3) in\ eq(2)$

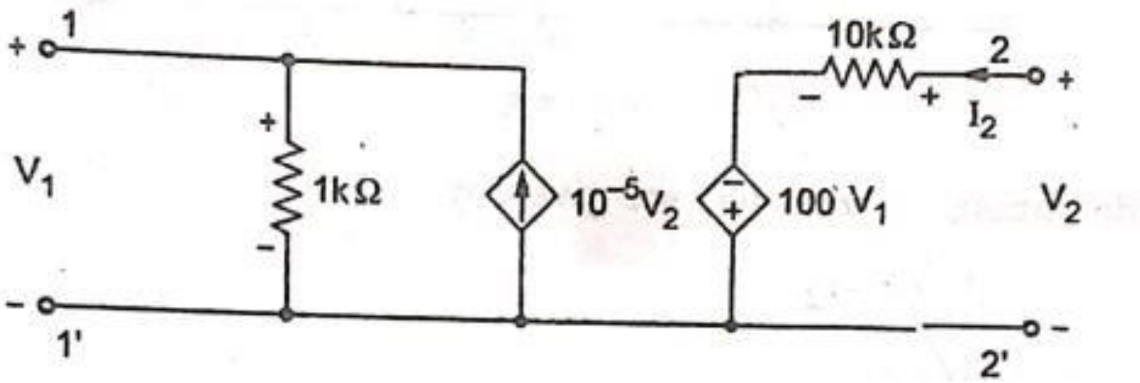
$$V_2 = -100 \left(500\ I_1\right)$$

$$V_2 = -50000\ I_1$$

$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = -50k\Omega$$

$V_1 = z_{11} I_1 + z_{12} I_2 \qquad V_2 = z_{21} I_1 + z_{22} I_2$

1. $Let\ I_1 = 0 \ \rightarrow port\ 1\ open\ circuited$



At Port 1

$$V_1 = (1 * 10^3)(10^{-5}\, V_2)) = 0.01\, V_2 \ldots (4)$$

Apply KVL at port 2

$$-10 * 10^3\, I_2 + 100\, V_1 + V_2 = 0$$

$$V_2 = 10 * 10^3\, I_2 - 100\, V_1 \ldots (5)$$

Substitute for V_1 from eq (4) in eq (5)

$$V_2 = 10 * 10^3\ I_2 - 100\,(0.01\, V_2)$$

$$V_2 = 10 * 10^3\ I_2 + V_2$$

$$2V_2 = 10 * 10^3\, I_2$$

$$V_2 = 5000\, I_2 \quad \ldots (6)$$

$$z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = 5000\ \Omega$$

Substitute for V_2 from eq (6) in eq (4)

$$V_1 = 0.01\,(5000\, I_2)$$

$$V_1 = 50 I_2$$

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \quad 50\ \Omega$$

$$z = \begin{bmatrix} 500 & 50 \\ -50000 & 5000 \end{bmatrix}$$

y-parameters from z-parameters

$$y = \begin{bmatrix} \frac{z_{22}}{\Delta z} & \frac{-z_{12}}{\Delta z} \\ \frac{-z_{21}}{\Delta z} & \frac{z_{11}}{\Delta z} \end{bmatrix}$$

$$\Delta z = 500 * 5000 - 50 * (-50000)$$
$$= 5 * 10^6$$

$$y = \begin{bmatrix} 1 * 10^{-3} & 1 * 10^{-5} \\ -10 * 10^{-3} & 1 * 10^{-4} \end{bmatrix}$$

Problem 2: The z-parameters of a two port network are $z_{11} = 20\Omega$, $Z_{22} = 30\Omega, z_{12} = z_{21} = 10\Omega$.
Find y and ABCD parameters of the network.

$$y = \begin{bmatrix} \frac{z_{22}}{\Delta z} & \frac{-z_{12}}{\Delta z} \\ \frac{-z_{21}}{\Delta z} & \frac{z_{11}}{\Delta z} \end{bmatrix}$$

$$[T] = \begin{bmatrix} \frac{z_{11}}{z_{21}} & \frac{\Delta z}{z_{21}} \\ \frac{1}{z_{21}} & \frac{z_{22}}{z_{21}} \end{bmatrix}$$

y-parameters

$$y_{11} = \frac{3}{50} \text{ u}$$

$$y_{12} = \frac{-1}{50} \text{ u}$$

$$y_{21} = \frac{-1}{50} \text{ u}$$

$$y_{22} = \frac{2}{50} \text{ u}$$

ABCD parameters

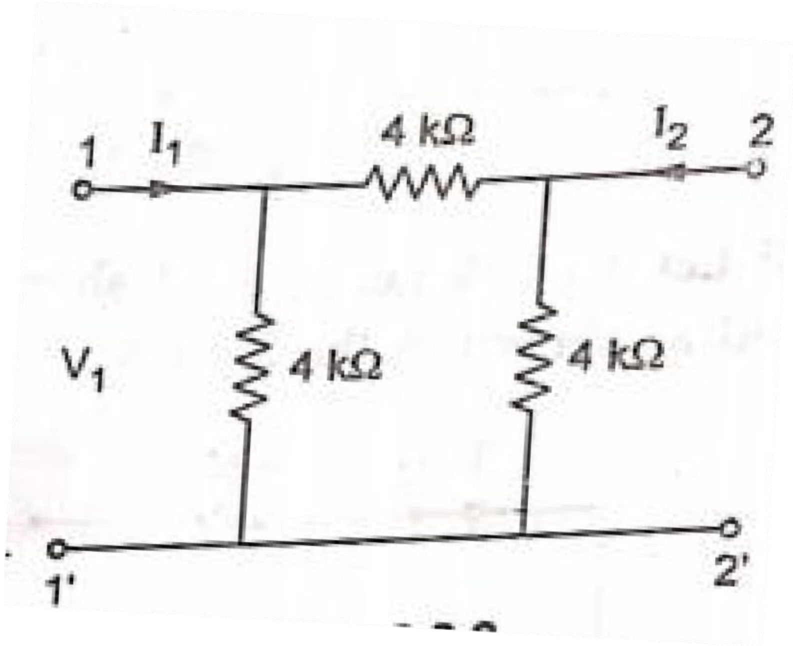
$$A = 2$$

$$\text{B} = 50\ \Omega$$

$$\text{C} = 0.1\text{u}$$

$$\text{D} = 3$$

Problem 3: Find the y-parameters for the circuit shown in figure. Then use the parameter relation ship to find the ABCD parameters. (assignment)



Solution

$$y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} 0.5 * 10^{-3} & -0.25 * 10^{-3} \\ -0.25 * 10^{-3} & 0.4 * 10^{-3} \end{bmatrix}$$

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 2 & 4000 \\ -0.75 * 10^{-3} & 2 \end{bmatrix}$$

Problem 4: Following are the hybrid parameters for a network.

h = $\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$

Determine the y-parameters of the network. (assignment)

Solution

y = $\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{3}{5} & \frac{24}{5} \end{bmatrix}$

MODULE-5
RESONANCE CIRCUITS

Contents

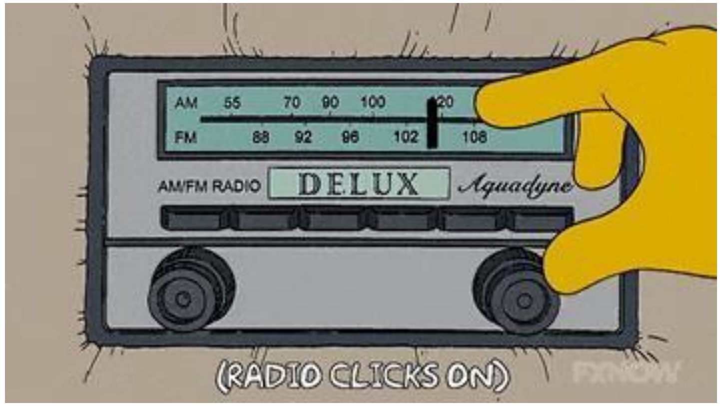
Series Resonance:

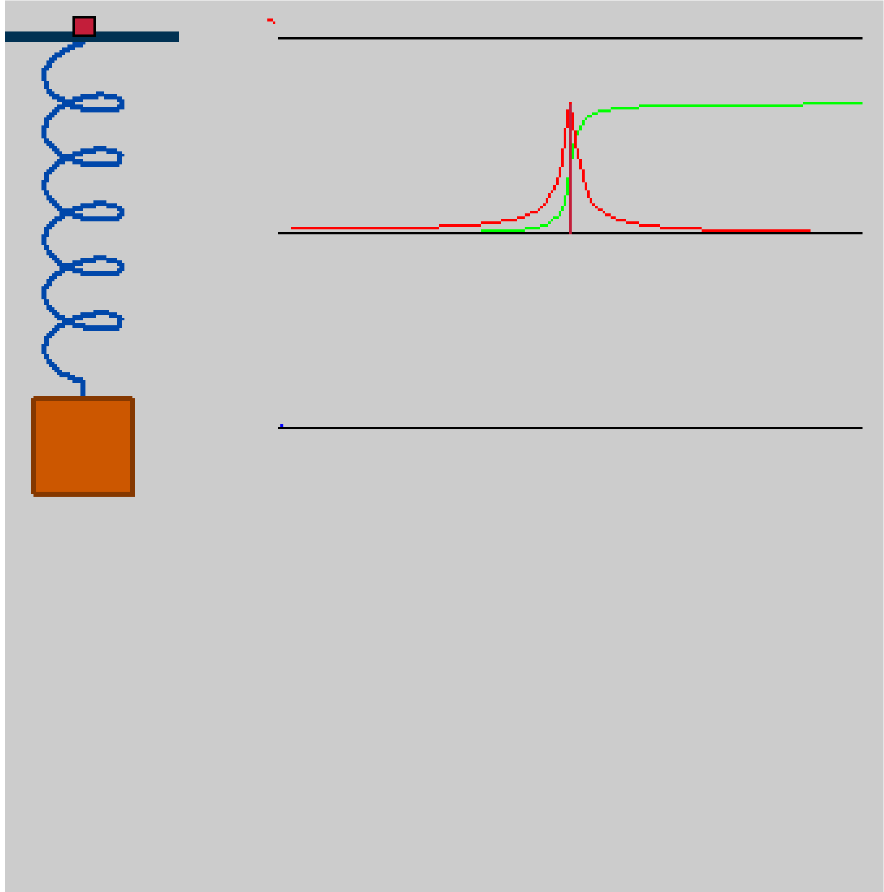
- Variation of current and voltage with frequency, selectivity and Bandwidth
- Q-factor, Circuit magnification factor
- Selectivity with variable capacitance, selectivity with variable inductance.

Parallel Resonance:

- Selectivity and Bandwidth
- Maximum impedance conditions with C, L & f variable
- Current in anti-resonant circuit
- The general case resistance present in both branches.

Introduction





Resonance

Resonance is defined as a phenomenon in which applied voltage and resulting current are in phase.

In AC circuits, under resonance condition, the reactance get cancelled, if the inductive and capacitive reactances are in series or the susceptance get cancelled if the inductive and capacitive reactances are in parallel.

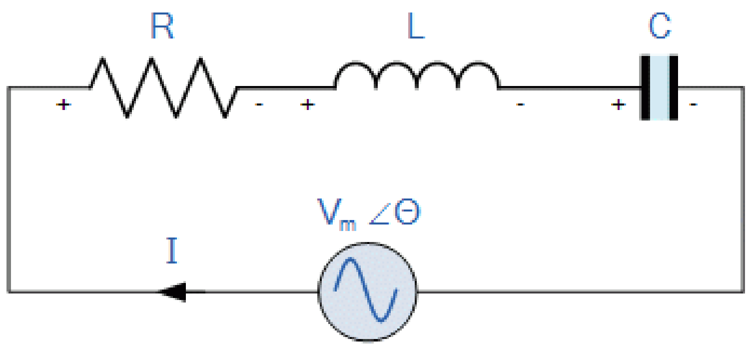
Complex impedance of AC circuit has only real resistance part.

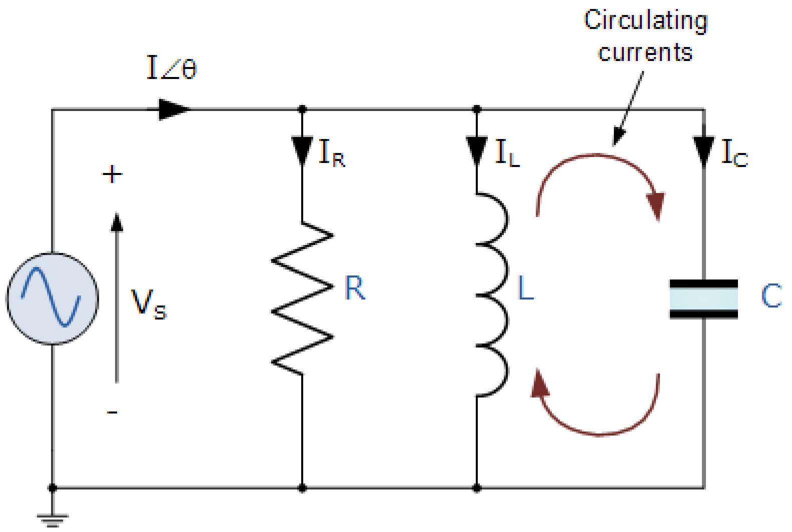
The resonance condition in AC circuits may be achieved by varying the frequency of the supply, keeping the network elements constant or by varying L or C, keeping the frequency constant.

Types of Resonance

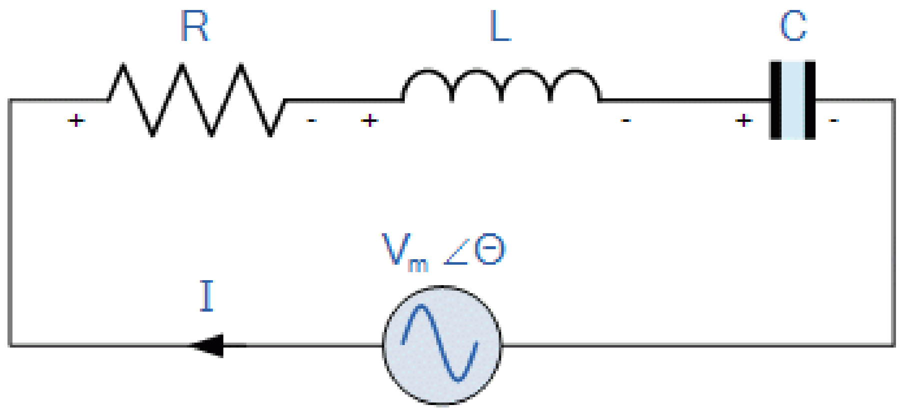
The resonance may be classified into two groups,

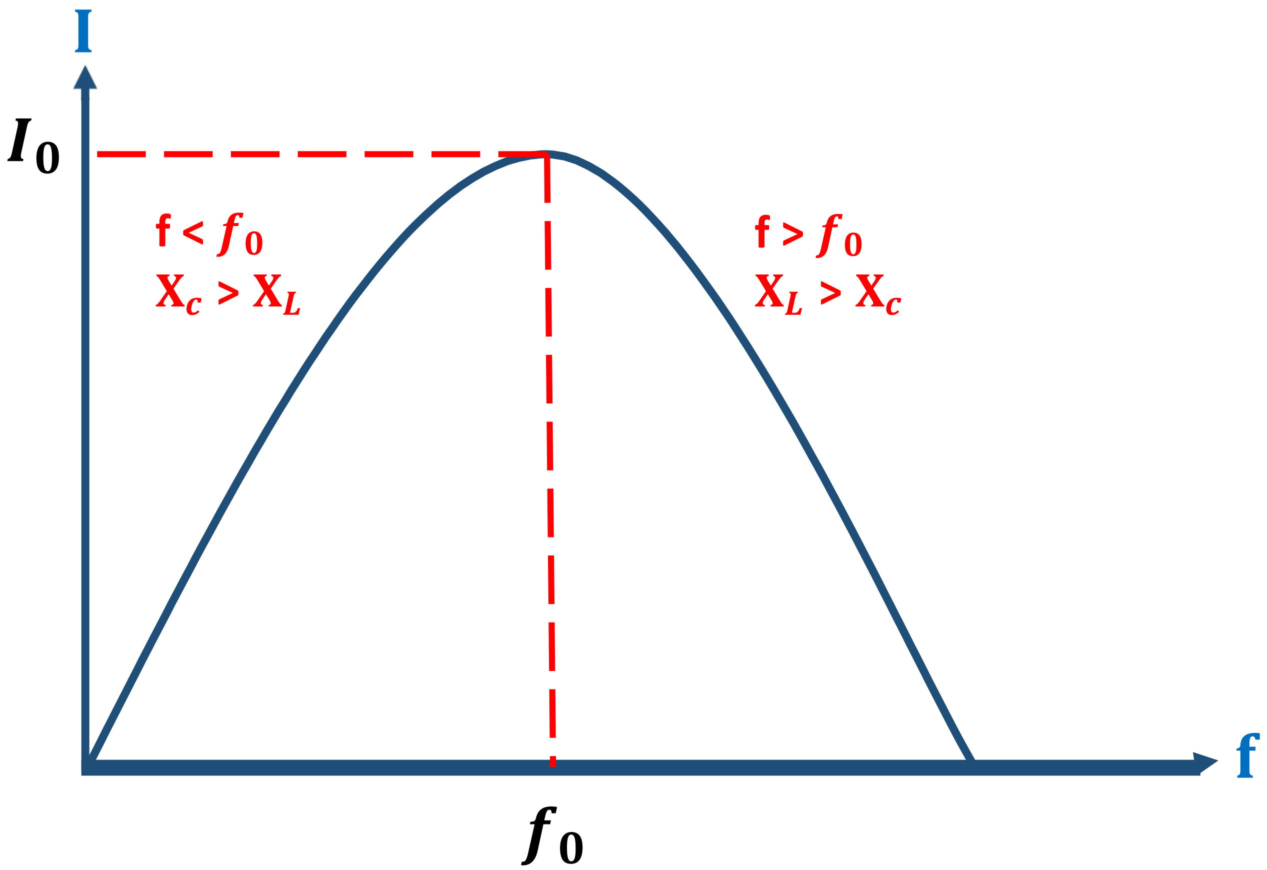
1. **Series resonance**
2. **Parallel resonance**





Series Resonance





Series Resonance

The impedance of the circuit is given by

$$Z = R + j\,(X_L - X_C)$$

$$Z = R + j\,\left(\omega L - \frac{1}{\omega C}\right)$$

According to the definition of resonance, reactive part of impedance of series RLC circuit is <u>zero</u>.

Let the frequency of resonance is denoted by ω_0.

$$\omega_0 L - \frac{1}{\omega_0 C} = 0$$

$$\omega_0 L = \frac{1}{\omega_0 C}$$

$$\omega_0^{\,2} = \frac{1}{LC}$$

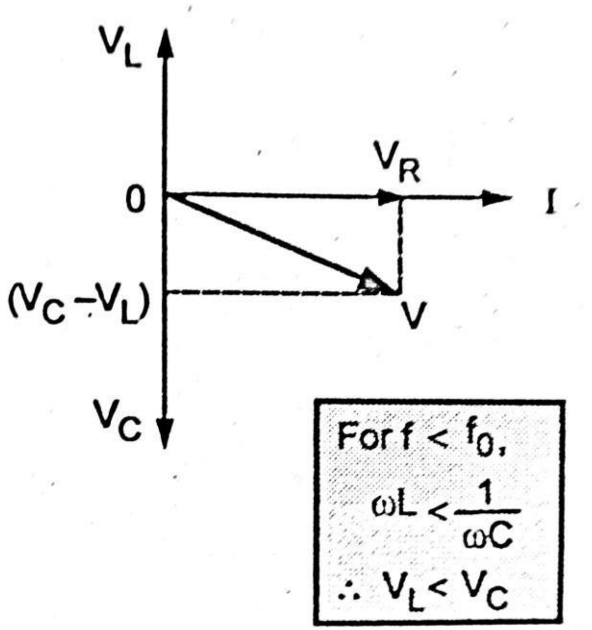
$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/sec}$$

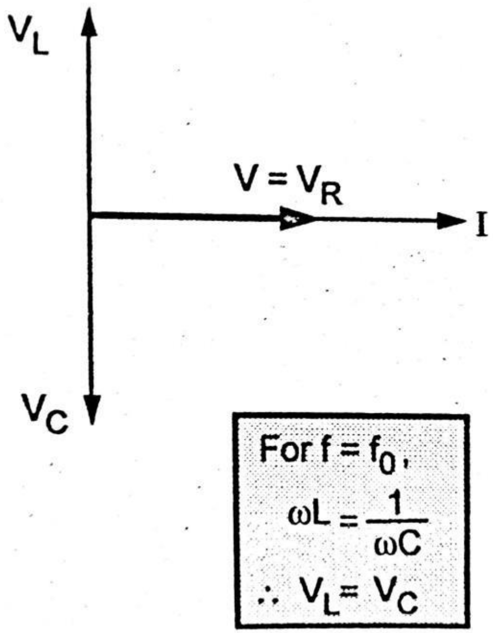
W.K.T, $\omega_0 = 2\pi f_0$

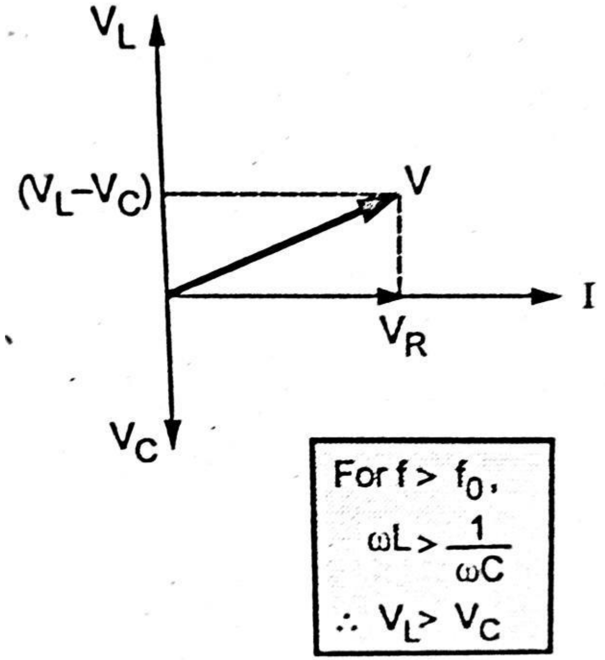
$$2\pi f_0 = \frac{1}{\sqrt{LC}}$$

$$\boxed{f_0 = \frac{1}{2\pi\sqrt{LC}}}$$

Phasor Diagram





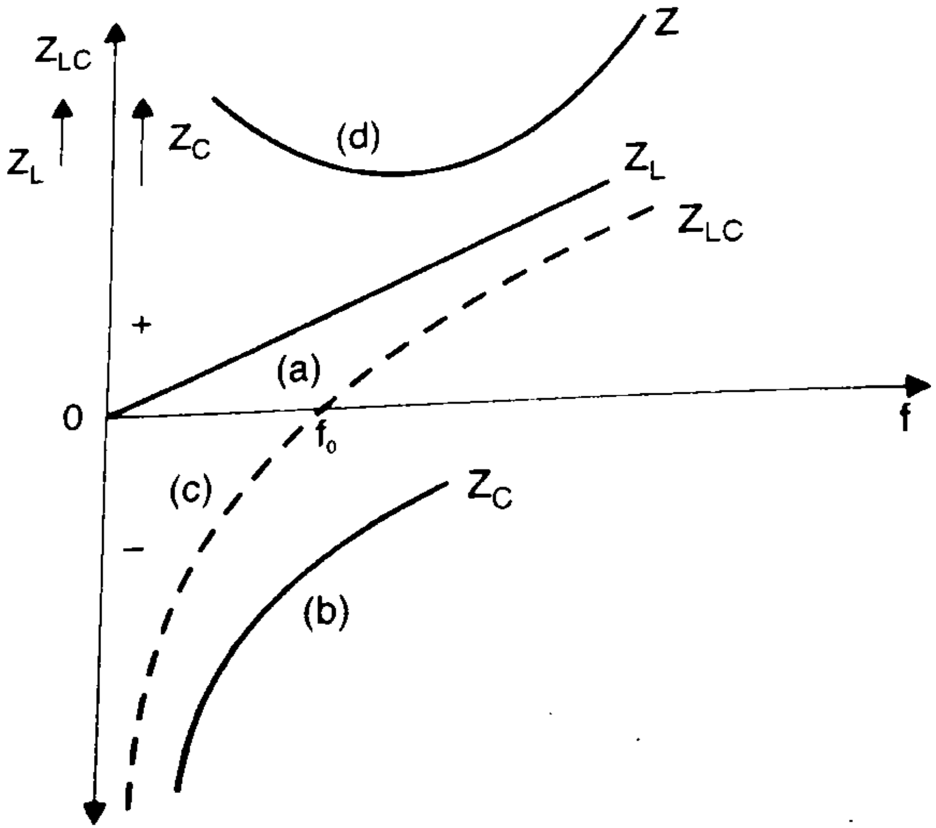


Reactance Curve

The impedance of the entire circuit

$Z = R + j\left(wL - \frac{1}{wC}\right)$

$|Z| = \sqrt{R^2 + \left(wL - \frac{1}{wC}\right)^2}$



Variation of current and voltage with frequency

The impedance of the circuit is

$$Z= R+j\left(wL - \frac{1}{wC}\right)$$

And the current is

$$I= \frac{V}{R+j\left(wL - \underline{wc}\,\right)}$$

which at resonance becomes $I_o = V/R$

Hence the current is maximum at resonance.

The voltage across capacitor C is

$$V_c = \frac{I}{jwC} = \frac{1}{jwC}\left[\frac{V}{R+j\left(wL - \frac{1}{wC}\right)}\right]$$

Hence , magnitude $\lvert V_c \rvert = \dfrac{V}{wC\sqrt{R^2 + \left(wL - \frac{1}{wC}\right)^2}}$

The frequency fc at which Vc is maximum may be obtained by equating $\frac{dVc^2}{dw}$ =0.

This results in fc = $\left(\frac{1}{2M}\right)\sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}$

The voltage across inductor L is

$$V_L = \frac{V(jwL)}{R+j\left(wL - \frac{}{wC}\right)}$$

The magnitude I V I = $\frac{VwL}{\sqrt{R^2 + \left(wL - \frac{1}{wC}\right)^2}}$

The frequency

f_L *at which* V_L is maximum may be obtained by equating $\frac{dV_L^2}{dw}$ *to zero.*

Thus, $f_L = \frac{1}{2\pi\sqrt{LC - \frac{C^2R^2}{2}}}$ obviously, $f_L > f_C$

The voltage V_L and Vc are equal in magnitude and opposite phase at resonance.

Bandwidth

Bandwidth of series RLC circuit is defined as the band of frequencies over which the power in the circuit is half of its maximum value.

At resonant frequency, maximum current I_0 is

$$I_0 = \frac{V}{R}$$

The current is maximum, as impedance is minimum at resonance.

$$P_0 = I_0^2 R = P_{max}$$

So, at the frequencies, where, power in the circuit is of its maximum value. Current becomes $(\frac{1}{\sqrt{2}})$ times of its maximum value.

"The frequencies at which power in the circuit is half of its maximum value is called half power frequencies."

At resonant frequency, power is given by,

$$P_0 = P_{max} = I_0^2 R$$

At f_1, power in the circuit is half.

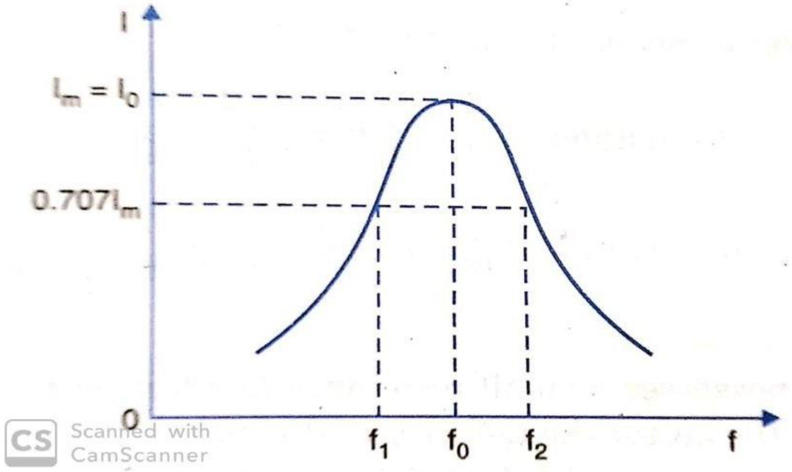
$$P' = \frac{1}{2} I_0^2 R$$

At f_2, power in the circuit is half.

$$P' = \frac{1}{2} I_0^2 R$$

According to definition,

$$B.W = (f_2 - f_2)\ Hz$$



Current in series RLC circuit is given by,

$$I = \frac{V}{|Z|} = \frac{V}{\sqrt{R^2 + (wL - \frac{1}{wC})^2}} \text{------------(1)}$$

At half power point,

$$I = \frac{I_0}{\sqrt{2}} \rightarrow \frac{1}{\sqrt{2}} \frac{V}{R} \text{------------------(2)}$$

Current in series RLC circuit is given by,

$$I = \frac{V}{|Z|} = \frac{V}{\sqrt{R^2 + (wL - \frac{1}{wC})^2}} \text{------------(1)}$$

At half power point,

$$I = \frac{I_0}{\sqrt{2}} \rightarrow \frac{1}{\sqrt{2}}\frac{V}{R} \text{-------------------(2)}$$

Equate (1) and (2)

$$\frac{1}{\sqrt{2}}\frac{V}{R} = \frac{V}{\sqrt{R^2 + (wL - \frac{1}{wC})^2}}$$

$$\sqrt{R^2 + (wL - \frac{1}{wC})^2} = \sqrt{2}\,R$$

Squaring on both side,

$$R^2 + (wL - \frac{1}{wC})^2 = 2\,R^2$$

$$(wL - \frac{1}{wC})^2 = R^2$$

$$\boxed{\pm R = wL - \frac{1}{wC}}$$

At half power frequencies f_1, f_2, the reactive part of impedance is equal to resistive part of impedance of RLC circuit.

$$\pm R = \omega L - \frac{1}{\omega C}$$

We can write,

$$\omega_2 L - \frac{1}{\omega_2 C} = +R \text{-----------------} (a)$$

$$\omega_1 L - \frac{1}{\omega_1 C} = -R \text{ ----------------- } (b)$$

(a) + (b) ➔

$$(\omega_1+\omega_2)L - \left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)\frac{1}{C} = 0$$

$$(\omega_1+\omega_2)L - \left(\frac{\omega_1+\omega_2}{\omega_1\omega_2}\right)\frac{1}{C} = 0$$

$$(\omega_1+\omega_2)L = \left(\frac{\omega_1+\omega_2}{\omega_1\omega}\right)\frac{1}{}$$

$$L = \left(\frac{1}{\omega_1\omega_2}\right)\frac{1}{C}$$

$$\omega_1\omega_2 = \frac{1}{LC} \text{ -----------------} (c)$$

But from the condition of resonance, $\omega_0 = \frac{1}{\sqrt{LC}}$ **rad/sec**

$$\omega_1\omega_2 = \omega_0^2$$

$$f_1 f_2 = f_0^2$$

$$\boxed{f_0 = \sqrt{f_1 f_2}}$$

$\omega_2 L - \frac{1}{\omega_2 C} = +R$ ----------------- (a)

$\omega_1 L - \frac{1}{\omega_1 C} = -R$ ----------------- (b)

$\omega_1 \omega_2 = \frac{1}{LC}$ ----------------- (c)

(a) - (b) ➔

$$(\omega_2 - \omega_1)L + \left(\frac{1}{\omega_1} - \frac{1}{\omega_2}\right)\frac{1}{C} = 2R$$

$$(\omega_2 - \omega_1)L + \left(\frac{\omega_2 - \omega_1}{\omega_1 \omega_2}\right)\frac{1}{C} = 2R$$

$$(\omega_2 - \omega_1) + \left(\frac{\omega_2 - \omega_1}{\omega_1 \omega_2}\right)\frac{1}{LC} = \frac{2R}{L}$$

$$(\omega_2 - \omega_1) + (\omega_2 - \omega_1)\frac{LC}{LC} = \frac{2R}{L}$$

$$2(\omega_2 - \omega_1) = \frac{2R}{L}$$

$$2\pi(f_2 - f_1) = \frac{R}{L}$$

$$\boxed{\text{B. W} = (f_2 - f_1) = \frac{R}{2\pi L}}$$

Selectivity

"Selectivity of a resonant circuit is defined as ability of a circuit to distinguish between desired & undesired frequency."

It is also the ratio of resonant frequency to the Bandwidth of a resonant circuit.

$$\text{Selectivity} = \frac{\text{Resonant frequency}}{\text{Bandwidth}} = \frac{f_0}{f_2 - f_1}$$

$$\text{Bandwidth} = f_2 - f_1 = \frac{R}{2\pi L}$$

$$\text{Selectivity} = \frac{f_0}{\frac{R}{2\pi L}} = \frac{2\pi f_0 L}{R}$$

$$= \frac{\omega_0 L}{R} = Q_0$$

Selectivity of resonant circuit is directly proportional to the Q-factor of the circuit at resonant frequency.

Q-factor

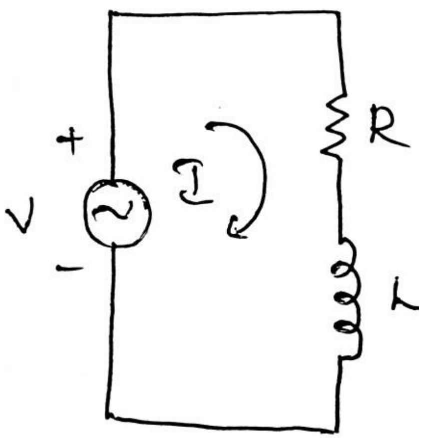
"Quality factor of a resonance circuit is a measure of quality of a resonant circuit."

Q is a ratio of power stored to the power dissipated in the circuit reactance & resistance respectively.

$$Q = 2\pi \times \frac{\text{Max energy stored by cycle}}{\text{Energy dissipated per cycle}}$$

Lets derive an expression for the Q-factor of inductor & Capacitor.

"Expression for Q-factor of an Inductor"



Consider a voltage V is applied to an inductor with leakage resistance R in series.

The maximum energy stored per cycle is given by

$$\omega_L = \frac{1}{2} L\, I_m^{\,2}$$

where $I_m \rightarrow$ Max current

The average power dissipated in inductor

$$P = \left(\frac{I_m}{\sqrt{2}}\right)^2 R \rightarrow \frac{1}{2} I_m^{\,2} R$$

Energy dissipated per cycle = $\frac{P}{f} = \frac{\frac{1}{2} I_m^{\,2} R}{f}$

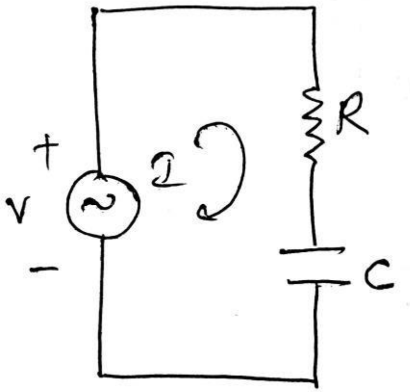
$f \rightarrow$ freq. of operation

$$Q = 2\pi \times \frac{\frac{1}{2} L\, I_m^{\,2}}{\frac{\frac{1}{2} I_m^{\,2} R}{f}} \rightarrow \frac{(2\pi f)\, L}{R}$$

$$\boxed{Q = \frac{\omega L}{R}}$$

"Expression for Q-factor of an Capacitor"

Consider a voltage V is applied to a capacitor with leakage resistance R in series.



The maximum energy stored per cycle is given by

$$\omega_C = \frac{1}{2} C\, {V_m}^2$$

where V_m ➔ Peak voltage

$$\omega_C = \frac{1}{2} C \left(\frac{I_m}{\omega C}\right)^2$$

$$\omega_C = \frac{1}{2} \frac{{I_m}^2}{\omega^2 C}$$

The average power dissipated in inductor

$$P = \left(\frac{I_m}{\sqrt{2}}\right)^2 R \rightarrow \frac{1}{2} {I_m}^2 R$$

Energy dissipated per cycle = $\frac{P}{f} = \frac{\frac{1}{2} {I_m}^2 R}{f}$

f ➔ freq. of operation

$$Q = 2\pi \times \frac{\frac{1}{2}\frac{{I_m}^2}{\omega^2 C}}{\frac{\frac{1}{2} {I_m}^2 R}{f}} \rightarrow \frac{(2\pi f)}{\omega^2 C R}$$

$$\boxed{Q = \frac{1}{\omega RC}}$$

"Expression for Q-factor of inductor & Capacitor"

Let $Q_0 = \frac{m_0 L}{R}$

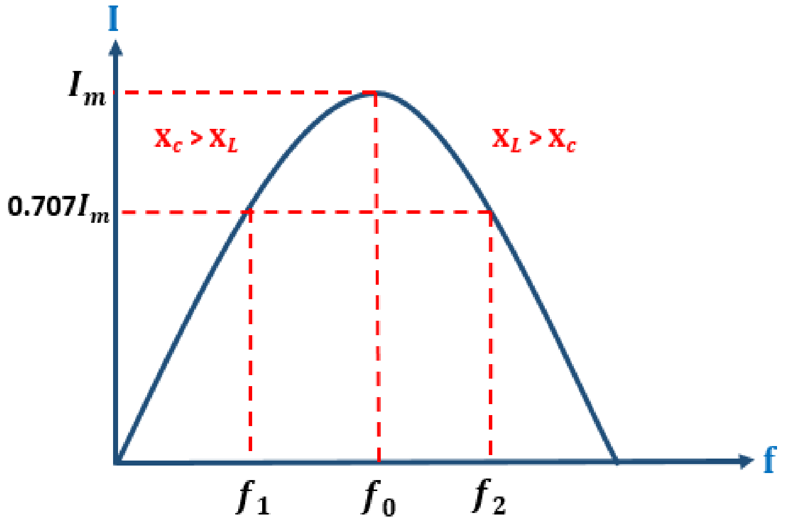
We know that, $m_0 = \frac{1}{\sqrt{LC}}$

$$Q_0 = \frac{\frac{1}{\sqrt{LC}} L}{R}$$

$$Q_0 = \sqrt{\frac{L}{C}}\ \frac{1}{R}$$

Expression for f_1 and f_2

f_1 and f_2 are equidistant from f_0 s shown in the figure.



At f_1 and f_2 , the current is $\frac{I_m}{\sqrt{2}}$ and hence, the impedance is $\sqrt{2}$ times the value of impedance.

At f_0, Z = R

At f_1 and f_2, Z = $\sqrt{2}$ R

In general, the impedance of circuit is given by,

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

At f_1 and f_2, $\sqrt{2}\,R = \sqrt{R^2 + (X_L - X_C)^2}$

$$2R^2 = R^2 + (\ - X_C)^2$$

$$R^2 = (X_L - X_C)^2$$

$$R = X_L - X_C \text{ ---------------------(*)}$$

$$R = X_L - X_C \text{ ----------------------(*)}$$

At f_1, $X_C > X_L$ Hence, eqn (*) can be written as

$$R = X_C - X_L$$

$$R = \frac{1}{\omega_1 C} - \omega_1 L$$

$$R = \frac{1 - \omega(\omega_1 L)}{\omega_1 C}$$

$$R = \frac{1 - \omega_1^2 CL}{\omega_1 C}$$

$$\omega_1 CR = 1 - \omega_1^2 CL$$

$$\omega_1 CR + \omega_1^2 CL - 1 = 0$$

[Divide by LC]

$$\omega_1^2 + \frac{\omega_1 R}{L} - \frac{1}{LC} = 0$$

{Quadratic equation} a=1, b=R/L, c= -1/LC

$$\omega_1 = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}}$$

$$f_1 = \frac{1}{2\pi}\left[-\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}}\right] \text{ ------------------------(1)}$$

$$R = X_L - X_C \text{ ---------------------(*)}$$

At f_2, $X_L > X_C$ Hence, eqn (*) can be written as

$$R = X_L - X_C$$

$$R = \omega_2 L - \frac{1}{\omega_2 C}$$

$$R = \frac{\omega_2 L(\omega_2 C) - 1}{\omega_1 C}$$

$$R = \frac{\omega_2{}^2 CL - 1}{\omega_2 C}$$

$$\omega_2 CR = \omega_2{}^2 CL - 1$$

$$\omega_2{}^2 CL - \omega_2 CR - 1 = 0$$ [Divide by LC]

$$\omega_2{}^2 - \frac{\omega_2 R}{L} - \frac{1}{LC} = 0$$ {Quadratic equation} a=1, b=-R/L, c= -1/LC

$$\omega_2 = \frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}}$$

$$f_2 = \frac{1}{2\pi}\left[\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}}\right] \text{ ----------------------(2)}$$

$$f_1 = \frac{1}{2\pi}\left[-\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}}\right] \text{------------------------}(1)$$

$$f_2 = \frac{1}{2\pi}\left[\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}}\right] \text{-------------------------}(2)$$

Bandwidth is given by $f_2 - f_1 = \frac{R}{2\pi L}$ (or) $\omega_2 - \omega_1 = \frac{R}{L}$ ------------(3)

But at resonance, $Q_0 = \frac{\omega_0 L}{R}$

$$Q_0 = \frac{2\pi f_0}{(R/L)}$$

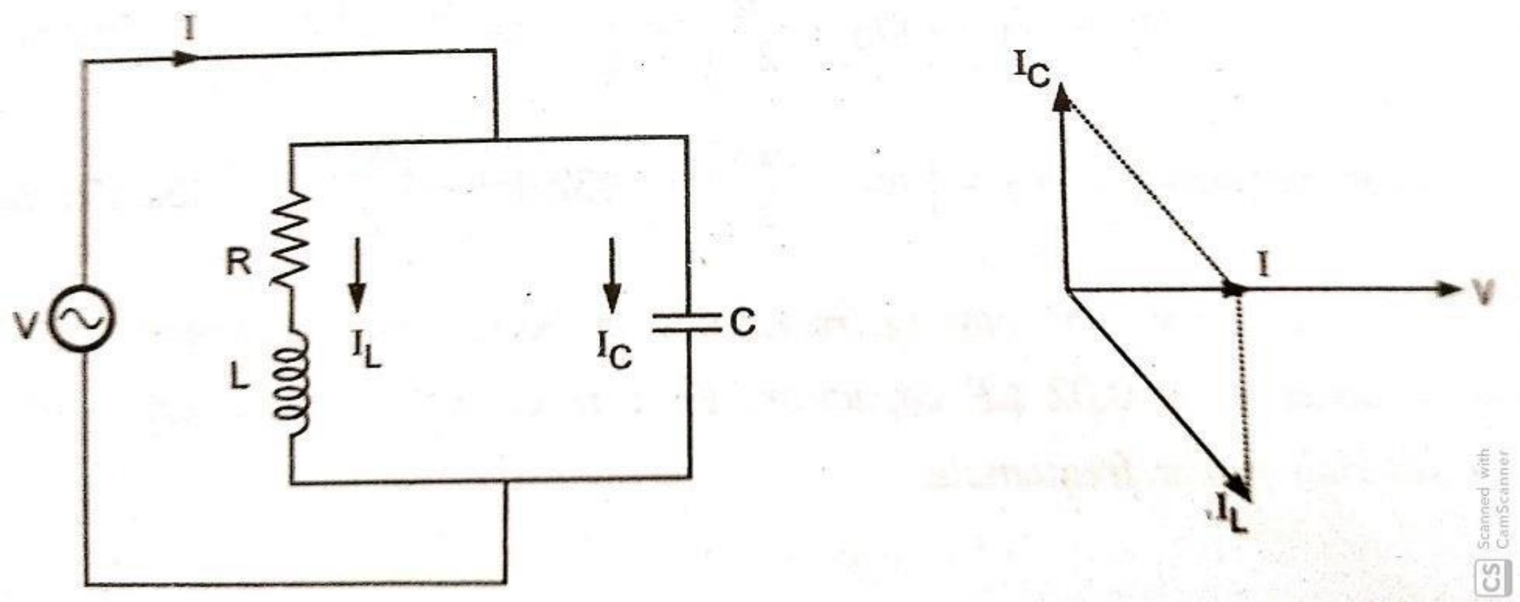
From eqn (3), $Q_0 = \frac{2\pi f_0}{2\pi(f_2 - f_1)}$

$$Q_0 = \frac{f_0}{B.W}$$

Parallel Resonance

A parallel circuit is said to be in resonance when applied voltage and resulting current are in phase that gives unity power factor condition.

Consider a parallel resonant circuit with applied voltage and total resulting current I.

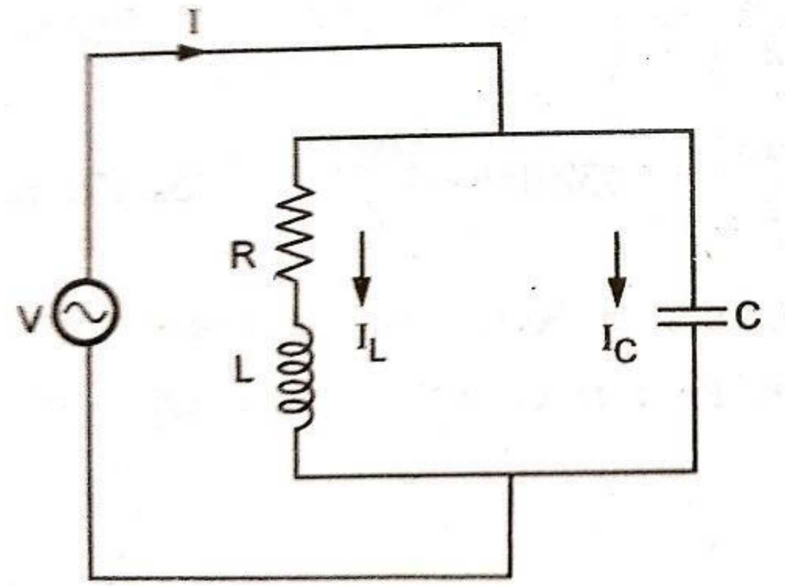


The admittance of branch containing L and R_L is

$$Y_L = \frac{1}{R + jX_L} \times \frac{R - jX_L}{R - jX_L}$$

$$= \frac{R - jX_L}{R^2 + X^2_L}$$

$$Y_L = \frac{R - jwL}{R^2 + w^2L^2}$$



The admittance of branch containing C is given by,

$$Y_C = \frac{1}{-jX_c} = j\frac{1}{X_c} = j.\frac{1}{1/wc} = jwc$$

Total admittance of parallel circuit is

$$Y = Y_L + Y_C$$

$$Y = \frac{R - jwL}{R^2 + w^2L^2} + jwc$$

$$Y = \frac{R}{R^2 + w^2L^2} + j\left[wc - \frac{wL}{R^2 + w^2L^2}\right] \text{----------------------------(1)}$$

At resonance , imaginary part i.e susceptance becomes zero.

Let the resonant frequency of parallel resonant circuit is denoted by w_{ar},

Thus at $w = w_{ar}$,

$$w_{ar}C - \frac{w_{ar}L}{R^2 + w_{ar}^2L^2} = 0$$

$$w_{ar}C = \frac{w_{ar}L}{R^2 + w_{ar}^2L^2}$$

$$R^2 + w_{ar}^2 L^2 = \frac{L}{C} \text{--------------------------(2)}$$

$$w_{ar}^2 L^2 = \frac{L}{C} - R^2$$

$$w_{ar}^2 = \frac{1}{LC} - \frac{R^2}{L^2}$$

$$w_{ar} = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

where w_{ar} **is the anti-resonant frequency**

$$2\pi f_{ar} = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$f_{ar} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}\sqrt{1 - \frac{CR^2}{L^2}}$$

Where f_{ar} is the series anti-resonance frequency.
For series resonance,

$$\omega_0 = \frac{1}{\sqrt{LC}} = 2\pi f_0$$

Where f_0 is the series resonant frequency

$$Q_0^2 = \frac{w_0 L}{R} \times \frac{1}{W_0 CR} = \frac{L}{CR^2}$$

$$f_{ar} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}\sqrt{1 - \frac{1}{Q_0^2}}$$

The impedance of the circuit can be obtained by putting susceptance part equal to zero in the expression of total admittance Y.

$$Y = \frac{1}{Z} = \frac{R_L}{R^2 + w^2 L^2}$$

$$Z = \frac{R^2 + w^2 L^2}{R}$$

From equation (2),

Hence the impedance at antiresonance is

$$Z_{ar} = \frac{L}{CR} = R_{ar}$$

From Equation 2

$$R^2 + w^2L^2 = \frac{L}{C}$$

$$R^2\left[1 + \frac{W^2L^2}{R^2}\right] = \frac{L}{C}$$

$$R^2\left[1 + Q_0^2\right] = \frac{L}{C}$$

$$R\left[1 + Q_0^2\right] = \frac{L}{CR}$$

$$Z_{ar} = R_{ar} = \frac{L}{CR} = R\left[1 + Q_0^2\right]$$

Impedance of Antiresonant Circuit Near Antiresonant

The impedance of parallel resonant circuit at any frequency is given by,

$$Z = (R_L + j\omega L) \parallel \left(\frac{1}{j\omega C}\right)$$

$$Z = \frac{(R_L + j\omega L)\left(\frac{1}{j\omega C}\right)}{R_L + j\omega L + \frac{1}{j\omega C}}$$

$$Z = \frac{R_L\left(1 + j\frac{\omega L}{R_L}\right)\left(\frac{1}{j\omega C}\right)}{R_L\left[1 + j\frac{\omega L}{R_L} + \frac{1}{j\omega R_L C}\right]}$$

$$Z = \frac{\left(1 + j\frac{\omega L}{R_L}\right)\left(\frac{1}{j\omega C}\right)}{1 + j\frac{\omega L}{R_L}\left(1 - \frac{1}{\omega^2 LC}\right)}$$

$$Z = \frac{\frac{L}{R_L C} - \frac{j}{\omega C}}{1 + j\frac{\omega L}{R_L}\left(1 - \frac{1}{\omega^2 LC}\right)} \quad \text{..............................(1)}$$

Above equation gives the general expression for the impedance of a parallel resonant circuit at any frequency w. Let δ be the fractional deviation in the frequency defined as,

$$\delta = \frac{f - f_{ar}}{f_{ar}} = \frac{\omega - \omega_{ar}}{\omega_{ar}} = \frac{\omega}{\omega_{ar}} - 1$$

$$\frac{\omega}{\omega_{ar}} = (1 + \delta) \quad \text{..(2)}$$

Also, ... $$\frac{\omega_{ar}}{\omega} = \frac{1}{(1+\delta)} \quad \text{..(3)}$$

We can write,

$$\frac{\omega L}{R_L} = \frac{\omega_{ar} L}{R_L} \cdot \frac{\omega}{\omega_{ar}}$$

$$\frac{\omega L}{R_L} = Q_0 \cdot (1 + \delta) \quad \text{...........................(4)}$$

Consider the expression

$$\frac{1}{\omega^2 LC} = \frac{\omega_{ar}^2}{\omega^2} \cdot \frac{1}{\omega_{ar}{}^2 LC}$$

But for high $Q_0, w_{ar} = \frac{1}{\sqrt{LC}}$ **therefore** $w_{ar}^2 = \frac{1}{LC}$ $i.e\ w_{ar}^2 LC = 1$

$$\frac{1}{\omega^2 LC} = \frac{\omega_{ar}^2}{\omega^2} \cdot 1 = \frac{\omega_{ar}^2}{\omega^2} = \frac{1}{(1+\delta)^2} \quad \text{.............................(5)}$$

Rearranging numerator term in Eq 1, we can write,

$$Z = \frac{\frac{L}{CR_L}\left(1+\frac{R_L}{j\omega L}\right)}{1+j\frac{\omega_L}{R_L}\left(1-\frac{1}{\omega^2 LC}\right)}$$

$$Z = \frac{\frac{L}{CR_L}\left(1-j\frac{R_L}{\omega L}\right)}{1+j\frac{\omega_L}{R_L}\left(1-\frac{1}{\omega^2 LC}\right)}$$

Putting values from equations 2,3,4 and 5 we get

$$Z = \frac{\frac{L}{CR_L}\left(1 - j\frac{R_L}{\omega L}\right)}{1 + j\frac{\omega_L}{R_L}\left(1 - \frac{1}{\omega^2 LC}\right)}$$

$$Z = \frac{L}{CR_L} \cdot \frac{1 - j\frac{1}{Q_0(1+\delta)}}{1 + jQ_0(1+\delta)\left[1 - \frac{1}{(1+\delta)^2}\right]}$$

$$Z = \frac{L}{CR_L} \cdot \frac{1 - j\frac{1}{Q_0(1+\delta)}}{1 + jQ_0(1+\delta)\left[\frac{(1+\delta)^2 - 1}{(1+\delta)^2}\right]}$$

$$Z = \frac{L}{CR_L} \cdot \frac{1 - j\frac{1}{Q_0(1+\delta)}}{1 + jQ_0\left[\frac{1 + 2\delta + \delta^2 - 1}{(1+\delta)}\right]}$$

$$Z = \frac{L}{CR_L} \cdot \frac{1 - j\frac{1}{Q_0(1+\delta)}}{1 + jQ_0 \cdot \delta\left[\frac{2+\delta}{1+\delta}\right]} \qquad \text{..............................(6)}$$

At resonance,

$$w = w_{ar}$$

$$\delta = \frac{\omega - \omega_{ar}}{\omega_{ar}} = 0$$

Then equation 6 reduces to,

$$Z = \frac{L}{CR_L} \cdot \left[1 - j\frac{1}{Q_0}\right]$$

But generally $Q_0 \gg 10, then\ \frac{1}{Q_0}$ is very small compared with unity term and hence usually neglected. Hence the impedance at antiresonance is given by

$$Z = Z_{ar} = \frac{L}{CR_L} \qquad \text{……………………(7)}$$

If $w \neq w_{ar}\ and\ \delta \ll 1, then\ neglecting\ \delta$ term as compared with unity term, then equation 6 reduces to

$$Z = \frac{L}{CR_L} \cdot \frac{1 - j\frac{1}{Q_0}}{1 + j2\delta Q_0}$$

With same justification as discussed earlier, $\frac{1}{Q_0} \ll 1$, hence neglected,

Therefore

$$Z = \frac{\frac{L}{CR_L} \cdot 1}{1 + j2\delta Q_0}$$

From equation 7,

$$\frac{L}{CR_L} = Z_{ar} = R_L (1 + Q_0^2) \approx R_L \cdot Q_0^2$$

Hence the impedance near resonance is given by

$$Z = \frac{Z_{ar}}{1 + j2\delta Q_0} = \frac{R_L \cdot {Q_0}^2}{1 + j2\delta Q_0}$$

Bandwidth and Selectivity of Antiresonant circuit

The half power frequency points f1 and f2 for parallel resonant circuit are obtained when impedance of the parallel resonant circuit Z becomes equal to (0.707) times value of maximum impedance at resonance Z_{ar}.

The impedance of parallel resonant circuit is given by

$$Z = \frac{R_L Q_0^2}{1 + j2\delta Q_0}$$

The impedance at antiresonance with high Q factor circuit is given by,

$$Z_{ar} = R_L Q_0^2$$

The condition for half power frequencies is given by,

$$\left|\frac{Z_{ar}}{Z}\right| = \sqrt{2}$$

$$|1 + j2\delta Q_0| = |1 \pm j1|$$

Comparing imaginary terms,

$$2\delta Q_0 = \pm 1 \delta$$

$$= \pm \frac{1}{2Q_0}$$

But δ is the fractional deviation and it is given by

$$\delta = \frac{f - f_{ar}}{f_{ar}} = \pm \frac{1}{2Q_0}$$

When frequency

$$f > f_{ar}, \quad \delta = +\frac{1}{2Q_0}$$

And

$$f < f_{ar}, \quad \delta = -\frac{1}{2Q_0}$$

Hence upper half power frequency is given by,

$$f_2 - f_{ar} = +\frac{f_{ar}}{2Q_0}$$

Similarly lower half power frequency is given by,

$$f_1 - f_{ar} = -\frac{f_{ar}}{2Q_0}$$

Therefore

$$\boxed{\text{Bandwidth} = f_2 - f_1 = \frac{f_{ar}}{Q_0}}$$

As we have studied in series resonance , the selectivity is given by,

$$\text{Selectivity} = \frac{\text{Resonant frequency}}{\text{Bandwidth}} = \frac{f_{ar}}{(f_2 - f_1)}$$

$$\boxed{\text{Selectivity} = \frac{f_{ar}}{(f_2 - f_1)} = Q_0}$$

Maximum Impedance Condition with C, L f Variables

The impedance of a parallel resonant circuit is

$$Z = \frac{(R + jX_L)(-jX_C)}{R + j(X_L - X_C)}$$

Where

$$X_L = \omega L,\ X_C = 1/\omega C$$

$$|Z|^2 = \frac{(R^2 + X_L^2)X_C^2}{R^2 + (X_L - X_C)^2}.$$

In order for $|z|^2$ *to be maximum by varying* X_c

$$\frac{d}{dX_C}|Z|^2 = 0$$

$$|Z|^2 = \frac{(R^2+X_L^2)X_C^2}{R^2+(X_L-X_C)^2}$$

$$\frac{[R^2+(X_L-X_C)^2]2X_C(R^2+X_L{}^2)-(R^2+X_L{}^2)X_C{}^2 2(X_L-X_C)(-1)}{\{R^2+(X_L-X_C)^2\}} = 0$$

$$[R^2+(X_L-X_C)^2]2X_C(R^2+X_L{}^2)+(R^2+X_L{}^2)X_C(X_L-X_C)=0$$

$$2X_C(R^2+X_L{}^2)\{R^2+(X_L-X_C)^2+X_C(X_L-X_C)\}=0$$

$$2X_C(R^2+X_L{}^2)\{R^2+X_L{}^2+X_C{}^2-2X_L+X_LX_C-X_C{}^2\}=0$$

$$2X_C(R^2+X_L{}^2)\{R^2+X_L{}^2-X_LX_C\}=0$$

$$\{R^2+X_L{}^2-X_LX_C\}=0$$

$$X_C=\frac{R^2+X_L^2}{X_L}=X_L\left[1+\left(\frac{R}{X_L}\right)^2\right]=X_L\left[1+\frac{1}{Q_0^2}\right]$$

This gives the value of the reactance X_C for maximum impedance.

Again $$R^2 + X_L^2 - X_C X_L = 0$$

$$R^2 + (wL) - \frac{1}{wC} wL = 0$$

$$R^2 + w^2 L^2 - \frac{}{wC} wL = 0$$

$$w^2 L^2 = \frac{L}{C} - R^2$$

$$w^2 = \frac{1}{CL} - \frac{R^2}{L^2}$$

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

Thus when capacitor C is varied for maximum impedance , the condition of unity power factor is automatically adjusted .
Maximum impedance by varying inductance can also be obtained by making

$$\frac{d}{dX_L}|Z|^2 = 0$$

Or

$$\frac{2[R^2 + (X_L - X_C)^2]\,X_L X_C^2 - 2(R^2 + X_L^2)\,X_C^2(X_L - X_C)}{[R^2 + (X_L - X_C)^2]^2}$$

Or

$$2X\{R^2 + (X_L - X_C)^2 X_L - (R^2 + X_L^{\,2})(X_L - X_C)\} = 0$$

$$2X\{[R^2 + (X_L - X_C)^2]X_L^2 - (R^2 + X_L^2)(X_L - X_C)\} = 0$$

$$2X_C^2\{R^2X_L + X_L^3 + X_C^2 X_L - 2X_L^2X_C - (R^2 X_L - R^2 X_C + X_L^3 - X_L^2X_C\} = 0$$

$$2X_C^2\{X_L + X_L^3 + X_C^2 X_L - 2X_L^2 X_C - R^2 X_L + R^2 X_C - X_L^3 + X_L^2 X_C\} = 0$$

$$2X\{X_C^2X_L - X_L^2X_C + R^2X_C\} = 0$$

$$X_C\{X_LX_C - X_L^2 + \} = 0$$

$$X_LX_C - X_L^2 + R^2 = 0$$

$$X_L^2 - X_LX_C - R^2 = 0$$

Solving above quadratic equation,

$$X_L = \frac{X_C}{2} + \sqrt{\left(\frac{X_C}{2}\right)^2 + R^2}$$

This gives the value of reactance X_L for maximum impedance and denotes the frequency at which it occurs as w_a.
Again from the equation 1

$$X_L^2 - X_L X_C - R^2 = 0$$

$$(\mathrm{wL})^2 - \frac{1}{wC}\mathrm{wL} - R^2 = 0$$

$$w^2\mathrm{L}^2 - \frac{}{wC}\mathrm{wL} - R^2 = 0$$

$$w^2\mathrm{L}^2 = \frac{L}{C} + R^2$$

$$w^2 = \frac{1}{LC} + \frac{R^2}{L^2}$$

$$\omega_{ar\,l} = \sqrt{\frac{1}{LC} + \frac{R^2}{L^2}}$$

With L adjusted for maximum impedance, $w_{ar_l} \neq w_{ar}$, where w_{ar} is the antiresonant frequency
Similarly , the maximum impedance condition can be obtained by varying frequency ,by

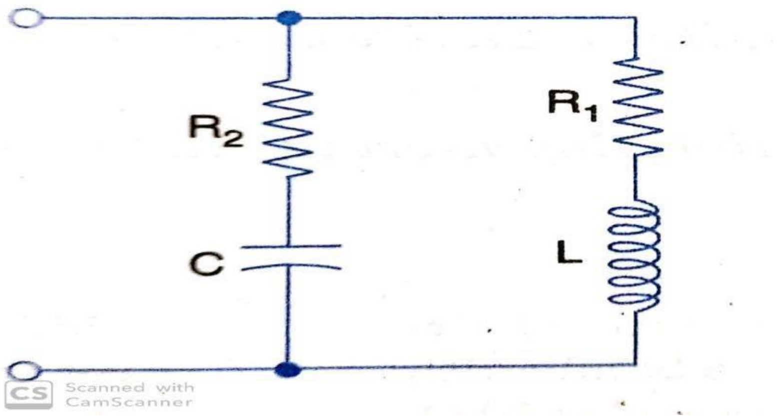
$$\frac{d}{d\omega}|Z|^2 = 0$$

Denoting the frequency at which it occurs as w_{arf}

$$\omega_{ar\,f} = \left[\frac{1}{LC}\sqrt{1 + \frac{\omega R^2 C}{L}} - \frac{R^2}{L^2}\right]^{1/2}$$

The General Case – Resistance Present in both Branches

In some types of anti-resonant circuits , a resistance may be present in series with the capacitive branch as well as the inductive branch, as shown in below figure



The admittance of the inductive branch is

$$Y_L = \frac{R_1 - j\omega L}{R_1^2 - \omega^2 L^2}$$

And that of capacitive branch is

$$Y_C = \frac{R_2 - j/\omega C}{R_2^2 - \dfrac{1}{\omega^2 C^2}}$$

Therefore , total admittance

$$Y = Y_L + Y_C = \frac{R_1}{R_1^2 + \omega^2 L^2} + \frac{R_2}{R_2^2 + \frac{1}{\omega^2 C^2}} - j\left(\frac{\omega L}{R_1^2 + \omega^2 L^2} + \frac{1/\omega C}{R_2^2 + \frac{1}{\omega^2 C^2}}\right)$$

For antiresonane, the reactive term must be zero, i.e

$$\omega_{ar} L\left(R_2^2 + \frac{1}{\omega_{ar}^2 C^2}\right) - \frac{1}{\omega_{ar} C^2}(R_1^2 + \omega_{ar}^2 L^2) = 0$$

Or

$$f_{ar} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}\left(\frac{L - R_1^2 C}{L - R_2^2 C}\right)}$$

Problem 1

A series RLC circuit consist of R=10Ω , L=0.01H, and C= 0.01μF is connected across a supply of 10mV. Determine , i) f_0 ii) Q-factor iii)BW iv) I_0

Solution:

i) $f_0 = \frac{1}{2\pi\sqrt{LC}}$

$=15.915\text{kHz}$

ii) Q-factor $Q_0 = \sqrt{\frac{L}{C}}\,\frac{1}{R}$

$= 100$

iii) B. W = $(f_2 - f_1) = \frac{R}{2\pi L}$

$= 159.155$

iv) $I_0 = \frac{V}{R}$

$= 1\text{mA}$

Problem 2

It is required that a series RLC circuit should resonate at 1 MHz. Determine values of R, L and C if bandwidth of the circuit is 5 kHz and its impedance is 50Ω at resonance.

Solution:

i) The impedance of series RLC circuit at resonance is defined as,

$$Z_0 = R$$

$$i.e\ R = 50\ \Omega$$

ii) B. W = $\frac{R}{2\pi L}$

= 5000

Therefore L = $\frac{R}{2\pi(B.W)}$

L = 1.5915mH

iii) The resonant frequency is given by,

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$C = \frac{1}{(2\pi f_0)^2(L)}$$

C= 15.9159pF

Problem 3

A coil is connected in series with a variable capacitor across v(t) = 10cos 1000t. The capacitor is varie and the current is maximum when C= 10μF. When C= 12.5μF , the current is 0.707 times the maximum value. Find L, R and Q of the coil.

Solution:

Comparing with standard expression for voltage

W = 1000 rad/sec

The current is maximum at C_0= 10μF

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$1000 = \frac{1}{\sqrt{LC_0}}$$

L= 0.1 H

At C= 12.5μF , current decreases to 0.707 times maximum current. Thus is half power condition . At half power condition , we can write

$$|X_L - X_c| = R$$

$$\left| W_0 L - \frac{1}{W_0 C} \right| = R$$

R= 20Ω

$$Q_0 = \frac{w_0 L}{R}$$

Q_0 = 5

Problem 4

A series RLC circuit consist of R=2Ω , L= 2 mH, and C= 10μF . Calculate Q factor , bandwidth, the resonant frequency and half power frequencies.

Solution:

i) Q-factor $Q_0 = \sqrt{\frac{L}{C}}\,\frac{1}{R}$

$= 7.071$

ii) B. W = $(f_2 - f_1) = \frac{R}{2\pi L}$

$= 159.155$ Hz

iii) The resonant frequency is given by

$$f_0 = (BW)(Q_0)$$

$= 1125.385$

iv) The resonant frequency is geometric mean of the half power frequencies i.e

$$f_0 = \sqrt{f_1 f_2} \text{ or } f_0^2 = f_1 f_2$$

Hence we can write two equations as,

B. W = $(f_2 - f_1) = 159.155$ Hz

And $f_1 f_2 = f_0^2 = (1125.385)^2 = 1.2665 * 10^6$

Putting value of f2 from equation 1 in equation 2 , we get

(159.155 + f_1)= $1.2665 * 10^6$

Therefore $f_1^2 + 159.155 f_1 - 1.2665 * 10^6$= 0

Solving quadratic equation for f_1 we get

$f_1 = 1.0486 * 10^3 Hz$ $\quad or \quad$ f_1= -1.2077kHz

Discarding f_1= -1.2077kHz as frequency can't be negative.

Hence

$$f_1 = 1.0486 * 10^3 Hz$$

Now $\quad f_2 - f_1 = 159.155$

$f_2 = 159.155 + f_1$

f_2= 1.2077kHz

Problem 5

An RLC series circuit has an inductive coil of 'R' Ω resistance and inductance of 'L' H is in series with a capacitor 'C' F. The circuit draws a maximum current of 15A when connected to 230V, 50Hz, supply. If the Q- factor is 5, find the parameter of the circuit.

Solution: $i_0 = 15A, V = 230V, f_0 = 50Hz, Q_0 = 5$

$I_0 = \frac{V}{R}$ i.e R= V/I_0

R= 15.33Ω

$f_0 = \frac{1}{2\pi\sqrt{LC}}$ i.e

LC= $1.0132*10^{-5}$..................................(1)

i) Q-factor $Q_0 = \sqrt{\frac{L}{C}}\frac{1}{R}$

$\frac{L}{C}$ = 5877.752 and use (1)

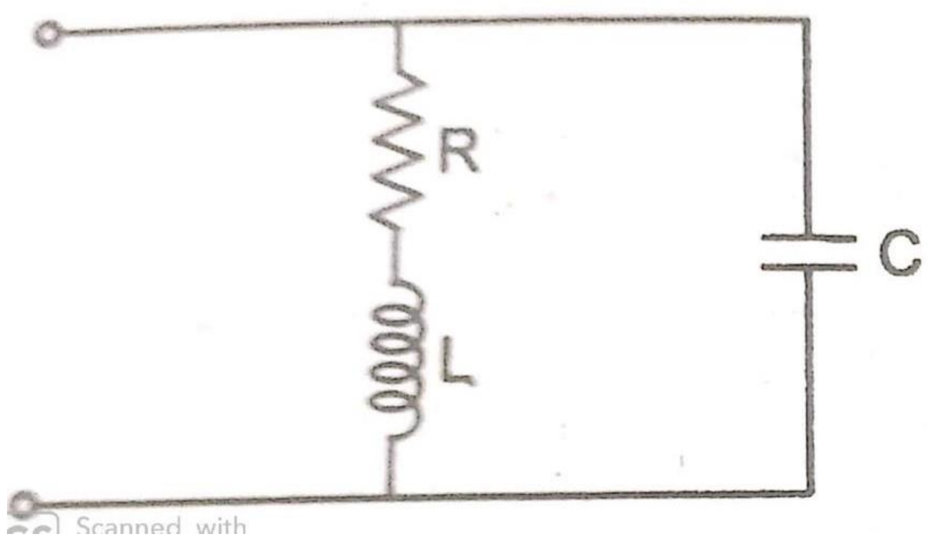
Therefore $\frac{L}{\left(\frac{1.0132*10^{-5}}{L}\right)}$ = 5877.752

L= 0.244H

C= 41.518μF

PROBLEMS ON PARALLEL RESONANCE

Problem 1: In the circuit given in fig, an inductance of 0.1H having a Q of 5 is in parallel with a capacitor. Determine the value of capacitance and coil resistance at resonance frequency of 500rad/sec. .



Solution: The given circuit is practical parallel resonant circuit. The antiresonant frequency interms of the Q- factor is given by

i) $$f_{ar} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}\sqrt{1-\frac{1}{Q_0^2}}$$

$$(2\pi f_{ar}) = \sqrt{\frac{1}{LC}}\sqrt{1-\frac{1}{Q_0^2}}$$

$$w_{ar} = 500 = \frac{1}{\sqrt{(0.1)(C)}}\sqrt{1-\frac{1}{(5)^2}}$$

C=38.4μF

The Q- factor at antiresonant frequency is given by

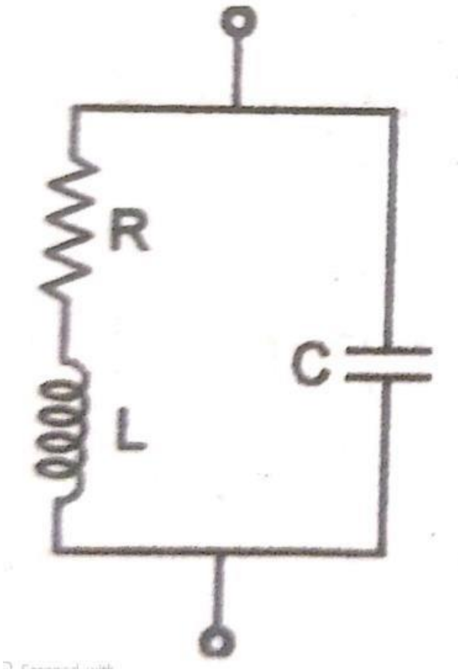
$$Q_0 = \frac{w_{ar}L}{R}$$

$$R = \frac{w_{ar}L}{Q_0}$$

R= 10Ω

Problem 2: If R= 25Ω , L= 0.5H and C=5μF, find the w_{ar}, Q and bandwidth for the circuit as shown in the figure

Solution:



i) $w_{ar} = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$

w_{ar} = 630.476 rad/sec

$$f_{ar} = \frac{w_{ar}}{2\pi}$$

f_{ar} =100.343Hz

ii) The Q- factor at antiresonant frequency is given by

$$Q_0 = \frac{w_{ar}L}{R}$$

Q_0= 12.6095

$$iv)\ BW = \frac{f_{ar}}{Q_0}$$

BW = 7.9577Hz

Problem 3: Find the value of R1 such that the circuit given in figure is resonant.

Solution:

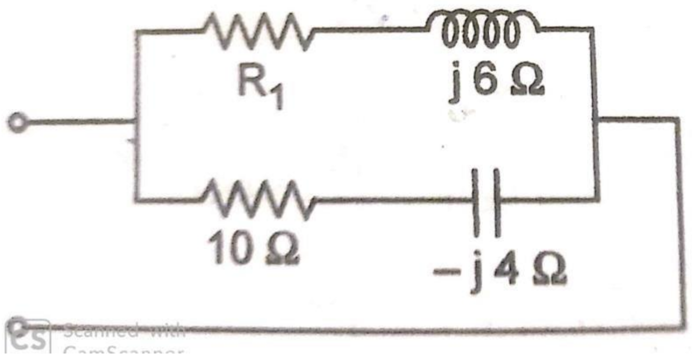
The total admittance of the circuit is given by,

$$Y_T = Y_L + Y_C = \frac{1}{Z_L} + \frac{1}{Z_c}$$

$$= \frac{1}{R_1 + j6} + \frac{1}{10 - j4}$$

$$= \frac{R_1 - j6}{R_1^2 + 36} + \frac{10 + j4}{100 + 16}$$

$$Y_T = \left[\frac{R_1}{R_1^2 + 36} + \frac{10}{116}\right] + j\left[\frac{4}{116} - \frac{6}{R_1^2 + 36}\right]$$



Now to have resonance in the parallel circuit, the susceptance should be zero.

Therefore $\frac{4}{116} - \frac{6}{R_1^2 + 36} = 0$

$$\frac{4}{116} = \frac{6}{R_1^2 + 36}$$

$R_1^2 + 36 = 176$

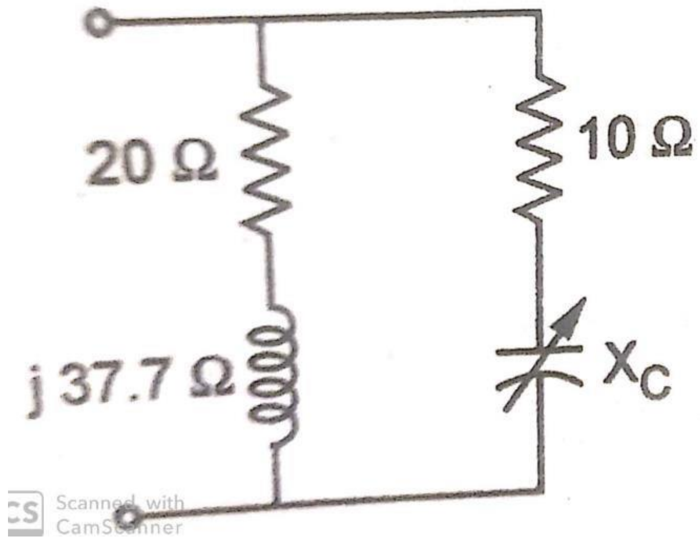
$R_1^2 = 138$

$R_1 = 11.7473\Omega$

Problem 4: For the circuit shown in figure, find two values of capacitor for the resonance. Derive the formula used. Consider f= 50Hz

Solution:

Let the resistance in series with inductance be R_L *and*
that in series with capacitance be R_C,
$\therefore R_L = 20\Omega\,, R_c = 10\Omega, jX_L = j37.7\Omega$
Total susceptance of parallel resonant circuit can be written as,



$$Y_T = Y_L + Y_C = \frac{1}{Z_L} + \frac{1}{Z_c}$$

$$=\frac{1}{R_L+jX_L} + \frac{1}{R_C-jX_C}$$

$$=\frac{R_L-jX_L}{R_L^2+X_L^2} + \frac{R_C+jX_C}{R_C^2+X_C^2}$$

$$Y_T = \left[\frac{R_L}{R_L^2+X_L^2} + \frac{R_C}{R_C^2+X_C^2}\right] + j\left[\frac{X_C}{R_C^2+X_C^2} - \frac{X_L}{R_L^2+X_L^2}\right]$$

Now at antiresonance , the susceptance is equal be zero. Hence equating imaginary term to zero , we get

$$\frac{X_C}{R_C^2+X_C^2} = \frac{X_L}{R_L^2+X_L^2}$$

i.e $\dfrac{X_C}{X_C^2+100} = \dfrac{37.7}{400+1421.49}$

$$\frac{X_C}{X_C^2 + 100} = \frac{37.7}{1821.29}$$

Simplifying expression,

$37.3(X_C^2 + 100)=1821.29X_C$

$X_C^2 + 100 = 48.31\ X_C$

$X_C^2 - 48.31\ X_C + 100 = 0$

Solving above quadratic equation, we get

$X_{C_1} = 46.1428$

Or $X_{C_2} = 2.1672$

But $X_C = \frac{1}{wC} = \frac{1}{2\pi f C}$

i.e $C = \frac{1}{2\pi f X_C} = \frac{1}{(100\pi)(X_c)}$

$C_1 = \frac{1}{(100\pi)(46.1428)}$ = 68.9836 μF and

$C_2 = \frac{1}{(100\pi)(2.1672)}$ = 1.4687 mF

Hence two values of capacitor to have resonance at 50Hz are 68.9836 μF and 1.4687 mF.